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CONTENTS

THE RANDOM SIEVE

by David Hawkins

Cramer (1) discusses a simple stochastic model for the distribution of primes. Let there be a sequence of independent trials of an event S_n with

$$Pr(S_n) = 1/\log n \tag{1}$$

The numbers P_1, P_2, ... P_m, ..., of which P_m is the mth value of n for which S_n occurs, will then with probability one have a limiting density $1/\log n$, like the primes. A number of other consclusions follow from the strong law of large numbers, for example that with probability one

$$\limsup_{m\to\infty} \frac{P_{m+1} - P_m}{\log^2 P_m} = 1 \tag{2}$$

The model is however quite artificial in that it has the prime number theorem built into it *ad hoc* and in its assumption of independence. Primes are not independent in a statistical sense. The occurrence of an unusually long run of composite numbers leads one to expect a compensating increase in the number of primes later on.

A more natural stochastic model is that of the random sieve. In the sieve of Eratosthenes we sieve out the multiples of every number which is not a multiple of some earlier sieving number. We define the random sieve as follows: Check the number 2, and then with probability ½ strike out each subsequent number. If P_2 is the first number not stricken out, check it and strike out each number thereafter with probability $1/P_2$. P_3 is the next number not stricken out, and we use $1/P_3$ as the probability with which to strike out each subsequent number, etc. The set of all possible sequences of numbers checked (sequences of random sieving numbers) contains the sequence of primes as one of its most probable members. Another typical member is the sequence of "lucky numbers" (2). S_n now stands for the proposition that n is a sieving number, and we have the following recurrence relation:

$$Pr(S_{n+1}) = Pr(S_n) - \frac{Pr(S_n)^2}{n} \tag{3}$$

Proof: Let T_n be the contradictory of S . Let S_n^* be the proposition that n is not sieved out by any sieving number less than $n - 1$, and let T_n^* be its contradictory. Then

$$T_{n+1} \equiv T_n T_{n+1} \vee S_n T^*_{n+1} \vee S_n T_{n+1} S^*_{n+1}$$

But

$$Pr(T_n T_{n+1}) = Pr(T_n)^2, \quad Pr(S_n T^*_{n+1}) = Pr(S_n) Pr(T_n),$$

and

$$Pr(S_n T_{n+1} S^*_{n+1}) = Pr(S_n)^2/n.$$

Since the alternatives are mutually exclusive, the **recurrence relation** follows.

We solve (3) as follows:

Substitute $g_n = 1/Pr(S_n)$, obtaining

$$g_{n+1} - g_n = \frac{1}{n - 1/g_n} \tag{4}$$

which may be solved recursively, using $g_n'' = \sum 1/n = L(n)$ as the first trial solution. This gives $g_n = L(n) + 0(1)$ and thus

$$Pr(S_n) \sim 1/\log n \tag{5}$$

Thus the random sieve gives asymptotically the result assumed *ad hoc* by Cramer. The events S are not, moreover, independent. In fact, it is obvious that

$$Pr(T_n T_{n+1} \ldots T_{n\ r-1}) = Pr(T_n)^r \tag{6}$$

which shows that the interdependence is negative, in conformity with our comment about the primes. The result does not affect the validity of (2), however, unless it strengthens it. For if in (6) we put $r = c(\log^2 n)$, it is easy to see that the occurrence of a run of that length without sieving number is asymptotically of probability $1/n^c$. From the convergence of $\sum 1/n^c$ for $c > 1$ it follows that with probability one the number of runs of such length is finite. Hence $\log^2 P_m$ is almost certainly an upper bound, from some P_m on, to the interval $P_{m+1} - P_m$. Because of interdependence it is more difficult to prove that this is false for any $c > 1$.

It is not difficult to define and solve various random-sieve problems analogous to those of multiplicative number theory, for example the relative frequency of pairs of sieving numbers separated by a given interval, the expected number of "divisors" of any number, etc.

In the case of the lucky numbers 1, 3, 7, 8, 13, 15, ..., we start with odd numbers and sieve out first every third odd number, leaving 7 as the next lucky. Then we sieve out every seventh of the remaining numbers, etc. In this case nothing is known about the asymptotic density, so we are in the position, say, of Gauss viz a viz the prime distribution. A randomization of this process is, however, just the random sieve again, and from this fact we immediately conjecture that the asymptotic density of luckies is $1/\log n$ rather than, say, $B/\log n$ where $B \neq 1$.*

There is one direction in which the random sieve may facilitate something more than conjectures. The distribution of numbers prime to the first m primes, or that of pairs of such numbers separated by a constant interval (e.g. 2) is in all likelihood more regular in a certain sense than the corresponding random sieve distribution, and if this is true it implies a number of results somewhat stronger than those that have been obtained, e.g. the infinity of twin primes.

Suppose that the first m random sieving numbers are the first m primes, $P_r = p_r$ $r = 1, 2, \dots m$. Then the distribution of the number of numbers not sieved out from a sequence of N consecutive numbers $> p_m$ is given by the binomial distribution, with probability $Q_m = (1 - 1/2)(1 - 1/3) \dots (1 - 1/p_m)$. This same distribution may be expressed in a different way as the sum of 2^m random variables (non-independent). Let $N(i)$ be the number of numbers sieved out by the sieving number p_i, $N(i,j)$ those sieved out both by p_i and p_j, etc. Those not sieved out will be given by the well-known combinatoric formula

$$N - \sum_i N(i) + \sum_{i,j} N(i,j) \dots + (-1)^m \; N(1,2,3,\dots m). \tag{7}$$

For the sieve of Eratosthenes, on the other hand, we have a precisely similar expression. In this case the distribution of numbers prime to $2.3.5.\,.\;\; p_m = K_m$ is periodic modulo K_m, and we pick our sequence of N numbers, $N << K_m$, at random from a period of length K_m. We can calculate moments for the distribution of the number given by (7), both for the random and the Eratosthenes case. If as seems likely we can prove that moments of even order for the Eratosthenes distribution are smaller than the corresponding moments of the random sieve distribution, then it will follow that the longest interval between numbers prime to K_m is of the order of $p_m \log p_m$, a stronger result than has been obtained by other methods. A similar argument applied to twins prime to K_m would, if valid, sstablish the infinity of twin primes. It is not hard to prove the inequality for second moments, but the problem of proof for moments of order $2k$ remains.

* * * * *

* Subsequently verified by W.E. Briggs and the author, and independently by P. Erdos. See "*The Lucky Number Theorem*" to be published in this magazine.

References:

(1) H. Cramer, *Acta Arithmetica*, Vol. 2, 1937, pp 23-8.

(2) Ulam *et al*, *On Certain sequences of Integers defined by Sieves* Mathematics Magazine Vol. 29 (1956) pp 117-122.

University of Colorado

A Slide-Bee

James G. Dyhikowski

Recently there appeared in this magazine an article entitled "Dig That Math." The article was humorous and somewhat sarcastic. It implied that mathematics would need a radical change before mathematics teams would be formed. I would like to quote an article datelined Chicago, March 19, 1957 and entitled, "Mathematics is a New 'Sport'."

"A group of engineers who are worried about the nation's shortage of engineers have devised a slide rule competition for youngsters. They hope it will 'make mathematics as popular as sports.'

"The first of a series of planned 'slide-bees' was held yesterday in suburban Wheaton High School. Seven-member teams from seven high competed, each armed with a slide rule.

"The contest was run off like a spelling bee, with contestants given mathematical problems and those answering them fastest piling up points towards the championship.

"Among questions posed was this one:

> How soon will the Russians overtake our supply of engineers, if they turn out 81,000 engineers every y ear to our 28,000 engineers? Assume that we now have 642,000 engineers, against 396,000 engineers in Russia, and that deaths and retirements are disregarded.

"The answer to this was 4 years and 7 months.

"The winner was George Guerin of suburban Hinsdale High School.

" Guerin's prize: A slide rule.

The only ambition which I have, is that this movement well spread over the whole country and get students interested in science and engineering.

CALCULATION OF A COMPLETE SYSTEM OF TENSORS WITH THE AID OF SYMBOLIC MULTIPLICATION

Lewis Bayard Robinson

Introduction

A tensor is a generalized covariant. Complete systems of covariants are given by the solutions of complete systems of linear partial differential equations.

Complete systems of tensors are given by the solutions of differential equations of the Riquier-Saltykov type which contain complete systems of linear partial differential equations as a special case.[1]

Most of us know that the German geometers write a binary form thus:

$$(ax)^2 \equiv (a_1x_1 + a_2x_2)(a_1x_1 + a_2x_2) \equiv a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 \qquad 2.$$

We shall use the above symbolic multiplication to help us integrate the differential equations which arise.

Dr. T.C. Doyle has written an important paper on the tensors of projective differential geometry.[3]

The author is extending the work of Wilczynski and his pupil E.B. Stouffer on the covariants of linear differential equations to tensors and like them he rests on the work of Lie but he uses the German symbolic notation as an auxiliary.

Professor Cramlet has written a work: *The Invariants of an N-ary Q-ic Differential Form.*[4] But his definition of a complete system of tensors differs altogether from that given by the author. Nor does he follow the method of Lie and he does not refer to Riquier. He bases his work on Aronhold. It is only remotely related to the work of the author.

The first paper published by the author on the above subject appeared in the Doklady of the Russian academy (1937). Professor Hlavaty of Praha wrote a review of this note and remarked that the author had not used a tensor notation. I have with considerable profit introduced a symbolic notation in this work.

1. See Riquier, *Les Systèmes D'Équations aux Derivées Partielles*, page 502
2. See Grace and Young, *Algebra of Invariants.*
3. Doyle, *Tensor Theory of Invariants for the Projective Differential Geometry of a Curved Surface.* Transactions of the American Mathematical Society, 1944.
4. See, *Annals of Mathematics*, 1930 page 134

Part (I)

In a previous work the author has computed a complete system of semitensors associated with the system of differential equations

$$(S)^{1.}\quad \begin{aligned} y_1'' + p_{11}y_1' + p_{12}y_2' + q_{11}y_1 + q_{12}y_2 &= 0 \\ y_2'' + p_{21}y_1' + p_{22}y_2' + q_{21}y_1 + q_{22}y_2 &= 0 \end{aligned}$$

In the following paper he will compute a complete system of tensors associated with the above system.

The tensors we shall consider are of order $r = 1$ when there is only one subscript. Such a tensor is I_i. I_{ij} is a tensor of order $r=2$.

We shall now set up the system of equations defining the tensors when $r = 2$. It will then be easy for anyone to write the system when $r = 1$.

When we were computing a complete system of semitensors the determinants of the transformation was

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}$$

The determinant of the transformation of our tensors is written

$$\Delta \equiv \begin{vmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & \bar{\xi}' \end{vmatrix}$$

The infinitesimal transformations of Δ are given by

$$\begin{vmatrix} 1+\phi_{11}\delta t & \phi_{12}\delta t & 0 \\ \phi_{21}\delta t & 1+\phi_{22}\delta t & 0 \\ 0 & 0 & 1+\xi'\delta t \end{vmatrix} = 1 + \phi_{11}\delta t + \phi_{22}\delta t + \xi'\delta t$$

The transformations of the tensors are written when $r=1$, thus:

$$\bar{I}_i = \sum_1^3 k \quad \Delta_{ki}\, I_k \qquad (i = 1,2,3)$$

When $r = 2$, they are written [2]

$$\bar{I}_i \cdot \bar{I}_j \equiv \bar{I}_{ij} = \sum_1^3 k \ \Delta_{ki} I_k \otimes \sum_1^3 l \ \Delta_{lj} I_l$$

1. See Robinson, *Un Système De Riquier et Le Calcul Tensoriel Bulletin de la Societe Mathematique*, 1940, page 129

2. The symbol $\otimes$ indicates symbolic multiplication and $\bar{I}_{ij} = \bar{I}_{ji}$

$$= \Delta_{1j}\Delta_{1i}I_{11} + (\Delta_{1j}\Delta_{2i} + \Delta_{2j}\Delta_{1i})I_{12} + (\Delta_{1j}\Delta_{3i} + \Delta_{3j}\Delta_{1i})I_{13}$$
$$+ \Delta_{2j}\Delta_{2i}I_{22} + (\Delta_{2j}\Delta_{3i} + \Delta_{3j}\Delta_{2i})I_{23} + \Delta_{3j}\Delta_{3i}I_{33}$$

We shall compute our tensors by Lie's method of infinitesimal transformations. The infinitesimal transformations are:

$$\delta I_{11} = + 2I_{11}\phi_{22}\delta t - 2I_{12}\phi_{12}\delta t + 2I_{11}\xi'\delta t$$

$$\delta I_{12} = - I_{22}\phi_{12}\delta t + I_{12}(\phi_{11} + \phi_{22})\delta t - I_{11}\phi_{21}\delta t + 2I_{12}\xi'\delta t$$

$$\delta I_{22} = + 2I_{22}\phi_{11}\delta t - 2I_{12}\phi_{21}\delta t + 2I_{22}\xi'\delta t$$

$$\delta I_{13} = - I_{23}\phi_{12}\delta t + I_{13}(\phi_{11} + 2\phi_{22})\delta t + I_{13}\xi'\delta t$$

$$\delta I_{23} = + I_{23}(2\phi_{11} + \phi_{22})\delta t - I_{13}\phi_{21}\delta t + I_{23}\xi'\delta t$$

$$\delta I_{33} = + 2I_{33}(\phi_{11} + \phi_{22})\delta t$$

And from the above follow the formulae denoted by (F).[1]

$$\Phi_{12}(I_{11})\phi_{12}\delta t + \Phi_{11}(I_{11})\phi_{11}\delta t + \Phi_{22}(I_{11})\phi_{22}\delta t + \Phi_{21}(I_{11})\phi_{21}\delta t$$
$$+ \Phi_{\xi'}(I_{11})\xi'\delta t = - 2I_{12}\phi_{12}\delta t + 2I_{11}\phi_{22}\delta t + 2I_{11}\xi'\delta t$$

$$\Phi_{12}(I_{12})\phi_{12}\delta t + \Phi_{11}(I_{12})\phi_{11}\delta t + \Phi_{22}(I_{12})\phi_{22}\delta t + \Phi_{21}(I_{12})\phi_{21}\delta t$$
$$+ \Phi_{\xi'}(I_{12})\xi'\delta t = - I_{22}\phi_{12}\delta t + I_{12}\phi_{11}\delta t + I_{12}\phi_{22}\delta t$$
$$- I_{11}\phi_{21}\delta t + 2I_{12}\xi'\delta t \qquad \text{et cetera.}$$

We write the equations analogous to system (A) in the anterior work

$$\Omega_{ij}(I_{kl}) = 0$$
$$\Psi_{ij}(I_{kl}) = 0$$
$$(i, j, k, l = 1, 2)$$

$\Phi_{12}(I_{11}) = - 2I_{12}$	$\Phi_{12}(I_{12}) = - I_{22}$
$\Phi_{11}(I_{11}) = 0$	$\Phi_{11}(I_{12}) = + I_{12}$
$\Phi_{22}(I_{11}) = + 2I_{11}$	$\Phi_{22}(I_{12}) = + I_{12}$
$\Phi_{21}(I_{11}) = 0$	$\Phi_{21}(I_{12}) = - I_{11}$

1. Wilczynski uses infinitesimal transformations to compute complete systems of covariants in his book *Projective Differential Geometry*.

$\Phi_{\xi'}(I_{11}) = +2I_{11}$ $\Phi_{\xi'}(I_{12}) = +2I_{12}$

$\Phi_{12}(I_{22}) = 0$ $\Phi_{12}(I_{13}) = -I_{23}$

$\Phi_{11}(I_{22}) = +2I_{22}$ $\Phi_{11}(I_{13}) = +I_{13}$

$\Phi_{22}(I_{22}) = 0$ $\Phi_{22}(I_{13}) = +2I_{13}$

$\Phi_{21}(I_{22}) = -2I_{12}$ $\Phi_{21}(I_{13}) = 0$

$\Phi_{\xi'}(I_{22}) = +2I_{22}$ $\Phi_{\xi'}(I_{13}) = +I_{13}$

$\Phi_{12}(I_{23}) = 0,$ $\Phi_{12}(I_{33}) = 0;$

$\Phi_{11}(I_{23}) = +2I_{23},$ $\Phi_{11}(I_{33}) = +2I_{33};$

$\Phi_{22}(I_{23}) = +I_{23},$ $\Phi_{22}(I_{33}) = +2I_{33};$

$\Phi_{21}(I_{23}) = -I_{13}$ $\Phi_{21}(I_{33}) = 0;$

$\Phi_{\xi'}(I_{23}) = +I_{23}$ $\Phi_{\xi'}(I_{33}) = 0.$

We derive the above from formulae (F) by equating the coefficients of the ϕ_{ij} and ξ'.

We know how to write the expressions, $\Omega_{ij}(F)$, $\Psi_{ij}(F)$, $\Phi_{ij}(F)$, for the author has given them in the anterior work. But we have introduced a new symbol $\Phi_{\xi'}(F)$, which we write

$$\Phi_{\xi'}(F) \equiv -y_1'\frac{\partial F}{\partial y_1'} - y_2'\frac{\partial F}{\partial y_2'} - p_{11}\frac{\partial F}{\partial p_{11}} - p_{12}\frac{\partial F}{\partial p_{12}}$$

$$-p_{21}\frac{\partial F}{\partial p_{21}} - p_{22}\frac{\partial F}{\partial p_{22}} - 4\Theta\frac{\partial F}{\partial \Theta}\ 5\Theta'\frac{\partial F}{\partial \Theta'}$$

where $\Theta \equiv \Theta_4 \equiv I^2 - 4J.$

Wilczynski has computed this seminvariant. See *Projective Differential Geometry*, page 110.

We observe that of the above equation Wilczynski uses only the terms

$$-4\Theta\frac{\partial F}{\partial \Theta} - 5\Theta'\frac{\partial F}{\partial \Theta'}$$

since he does not for his purposes require the other terms, **when** he is computing only invariants.[1]

And to the equations of the above system we add one more set:

$$\Phi_{\xi''}(I_{kl}) \equiv -y_1' \frac{\partial I_{kl}}{\partial y_1''} - y_2' \frac{\partial I_{kl}}{\partial y_2''} + \frac{\partial I_{kl}}{\partial p_{11}} + \frac{\partial I_{kl}}{\partial p_{22}} - 4\Theta \frac{\partial I_{kl}}{\partial \Theta'} = 0.$$ [2]

All the equations of the above system we will denote by (A).

Since this system (A) *represents the infinitesimal transformations of the* $\overline{I}_{kl}$ *the totality of all the solutions of* (A) *gives us the complete system of tensors.*

Now let

$$Z_1 \equiv 2y_1' + y_1 p_{11} + y_2 p_{12}$$

$$Z_2 \equiv 2y_2' + y_1 p_{21} + y_2 p_{22}$$

Substituting these new variables in $\Phi_{\xi'}(F)$ and $\Phi_{\xi''}(F)$ we obtain

$$\Phi_{\xi'}(F) \equiv -Z_1 \frac{\partial F}{\partial F_1} - Z_2 \frac{\partial F}{\partial Z_2} - 4\Theta \frac{\partial F}{\partial \Theta} - 5\Theta' \frac{\partial F}{\partial \Theta'} = 0$$

$$\Phi_{\xi''}(F) \equiv y_1 \frac{\partial F}{\partial Z_1} + y_2 \frac{\partial F}{\partial Z_2} - 4\Theta \frac{\partial F}{\partial \Theta'} = 0$$

These are the equations which correspond to *system* (A) in the anterior work. Let us now set up the equations which correspond to system (B) in the anterior work. We write:

$$(B) \qquad \begin{aligned} \Omega_{ij}(F) &= 0 \\ \Psi_{ij}(F) &= 0 \\ \Phi_{\xi''}(F) &= 0 \end{aligned}$$

$$\Phi_{\xi'}(F) + 2I_{11} \frac{\partial F}{\partial I_{11}} + 2I_{12} \frac{\partial F}{\partial I_{12}} + 2I_{22} \frac{\partial F}{\partial I_{22}} + I_{13} \frac{\partial F}{\partial I_{13}} + I_{23} \frac{\partial F}{\partial I_{23}} = 0$$

$$\Phi_{12}(F) - 2I_{12} \frac{\partial F}{\partial I_{11}} - I_{22} \frac{\partial F}{\partial I_{12}} - I_{23} \frac{\partial F}{\partial I_{13}} = 0$$

$$\Phi_{11}(F) + I_{12} \frac{\partial F}{\partial I_{12}} + 2I_{22} \frac{\partial F}{\partial I_{22}} + I_{13} \frac{\partial F}{\partial I_{13}} + 2I_{23} \frac{\partial F}{\partial I_{23}} + 2I_{33} \frac{\partial F}{\partial I_{33}} = 0$$

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1. See *Projective Differential Geometry*, page 121, formula (103). We replace Θ_4 by Θ
2. See the just cited formula. (103).

$$\Phi_{33}(F) + 2I_{11}\frac{\partial F}{\partial I_{11}} + I_{12}\frac{\partial F}{\partial I_{12}} + 2I_{13}\frac{\partial F}{\partial I_{13}} + I_{23}\frac{\partial F}{\partial I_{23}} + 2I_{33}\frac{\partial F}{\partial I_{33}} = 0$$

$$\Phi_{21}(F) - I_{11}\frac{\partial F}{\partial I_{12}} - 2I_{12}\frac{\partial F}{\partial I_{22}} - I_{13}\frac{\partial F}{\partial I_{23}} = 0$$

Write the new variables

$$Y_1 \equiv Z_1 + \frac{y_1\Theta'}{4\Theta}$$

$$Y_2 \equiv Z_2 + \frac{y_2\Theta'}{4\Theta}$$

After substituting these new variables in the above equations we obtain

$$-y_2\frac{\partial F}{\partial y_1} - Y_2\frac{\partial F}{\partial Y_1} - 2I_{12}\frac{\partial F}{\partial I_{11}} - I_{22}\frac{\partial F}{\partial I_{12}} - I_{23}\frac{\partial F}{\partial I_{13}} = 0$$

(I)[1]

$$-y_1\frac{\partial F}{\partial y_1} - Y_1\frac{\partial F}{\partial Y_1} + I_{12}\frac{\partial F}{\partial I_{12}} + 2I_{22}\frac{\partial F}{\partial I_{22}} + I_{13}\frac{\partial F}{\partial I_{13}} + 2I_{23}\frac{\partial F}{\partial I_{23}}$$

$$2I_{33}\frac{\partial F}{\partial I_{33}} = 0$$

$$-y_2\frac{\partial F}{\partial y_2} - Y_2\frac{\partial F}{\partial Y_2} + 2I_{11}\frac{\partial F}{\partial I_{11}} + I_{12}\frac{\partial F}{\partial I_{12}} + 2I_{13}\frac{\partial F}{\partial I_{13}} + I_{23}\frac{\partial F}{\partial I_{23}}$$

$$+ 2I_{33}\frac{\partial F}{\partial I_{33}} = 0$$

$$-y_1\frac{\partial F}{\partial y_2} - Y_1\frac{\partial F}{\partial Y_2} - I_{11}\frac{\partial F}{\partial I_{12}} - 2I_{12}\frac{\partial F}{\partial I_{22}} - I_{13}\frac{\partial F}{\partial I_{23}} = 0$$

$$-4\Theta\frac{\partial F}{\partial \Theta} - Y_1\frac{\partial F}{\partial Y_1} - Y_2\frac{\partial F}{\partial Y_2} + I_{11}\frac{\partial F}{\partial I_{11}} + 2I_{12}\frac{\partial F}{\partial I_{12}} + 2I_{22}\frac{\partial F}{\partial I_{22}}$$

$$+ I_{13}\frac{\partial F}{\partial I_{13}} + I_{23}\frac{\partial F}{\partial I_{23}} = 0$$

1. (I) indicates all five equations.

From the above five equations we have omitted the differential coefficients $\frac{\partial F}{\partial p_{ij}}, \frac{\partial F}{\partial p'_{ij}}, \frac{\partial F}{\partial q_{ij}}$, because we do not need them to compute tensors. (I) is a reduced form of (B).

Also Y_1 and Y_2 satisfy

$$\Omega_{ij}(F) = 0$$

$$\Psi_{ij}(F) = 0$$

$$\Phi_{\xi''}(F) = 0$$

which therefore vanish identically and appear no more in this work.

If we can solve the five key equations (I) *we can compute the complete system of tensors.*

Part (II).

We shall begin by writing the five key equations for the case where $r = 1$.

When we have solved these we can solve (II) by symbolic multiplication.

Lastly on page 14 we will write the formula which gives the complete system of tensors.

The five key equations are

$$- y_2 \frac{\partial F}{\partial y_1} - Y_2 \frac{\partial F}{\partial Y_1} - I_2 \frac{\partial F}{\partial I_1} = 0$$

$$(II) \quad \begin{aligned} & - y_1 \frac{\partial F}{\partial y_1} - Y_1 \frac{\partial F}{\partial Y_1} + I_2 \frac{\partial F}{\partial I_2} + I_3 \frac{\partial F}{\partial I_3} = 0 \\ & - y_2 \frac{\partial F}{\partial y_2} - Y_2 \frac{\partial F}{\partial Y_2} + I_1 \frac{\partial F}{\partial I_1} + I_3 \frac{\partial F}{\partial I_3} = 0 \end{aligned}$$

$$- y_1 \frac{\partial F}{\partial y_2} - Y_1 \frac{\partial F}{\partial Y_2} - I_1 \frac{\partial F}{\partial I_2} = 0$$

$$- 4 \Theta \frac{\partial F}{\partial \Theta} - Y_1 \frac{\partial F}{\partial Y_1} - Y_2 \frac{\partial F}{\partial Y_2} + I_1 \frac{\partial F}{\partial I_1} + I_2 \frac{\partial F}{\partial I_2} = 0 \text{ [1]}$$

1. (II) denotes the five equations.

The fundamental solutions of (II) are written

$$F_1 \equiv \Theta^{\frac{1}{4}}\{y_1 I_2 - y_2 I_1\}$$

$$F_2 \equiv Y_1 I_2 - Y_2 I_1$$

$$F_3 \equiv \Theta^{-\frac{1}{4}} \begin{vmatrix} y_1 & Y_1 \\ y_2 & Y_2 \end{vmatrix} I_3 \equiv \Theta^{-\frac{1}{4}} D\, I_3$$

$$\Theta \equiv \Theta_4 \equiv I^2 - 4J$$

as we have already stated. (See pages 8 and 9.

Now

$$(\bar{y}_1 \bar{I}_2 - \bar{y}_2 \bar{I}_1)^2 = \bar{y}_1^2\, \bar{I}_2 \cdot \bar{I}_2 - 2\, \bar{y}_1 \bar{y}_2 \bar{I}_1 \cdot \bar{I}_2 + \bar{y}_2^2\, \bar{I}_1 \cdot \bar{I}_1$$

$$= y_1^2\, I_2 \cdot I_2 - 2\, y_1 y_2 I_1 \cdot I_2 + y_2^2 I_1 \cdot I_1$$

The above is unaltered by all transformations of

$$I_1 \cdot I_1 \qquad I_1 \cdot I_2 \qquad I_2 \cdot I_2 .$$

But referring to foot note page 6, we replace $I_i \cdot I_j$ *by* I_{ij} and write

$$\bar{\Theta}^{\frac{1}{2}}\, (\bar{y}_1 \bar{I}_2 - \bar{y}_2 \bar{I}_1)^2 = \bar{\Theta}^{\frac{1}{2}}\{\bar{y}_1^2\, \bar{I}_{22} - 2\bar{y}_1 \bar{y}_2 \bar{I}_{12} + \bar{y}_2^2\, \bar{I}_{11}\} =$$

$$\Theta^{\frac{1}{2}}\, \{y_1^2\, I_{22} - 2y_1 y_2 I_{12} + y_2^2\, I_{11}\}$$

We perceive the above is unaltered by the transformations of

$$I_{11}, \quad I_{12}, \quad I_{22}.$$

so it is a solution of (I).

In a similar way we get other solutions of (I).

In short the transformations of $I_1 \cdot I \quad I_1 \cdot I_2 \quad I_2 \cdot I_2$ and $I_{11} \quad I_{12} \quad I_{22}$ *are isomorphic.*

To verify the above statements we need only write

$$\bar{I}_1 \cdot \bar{I}_1 = \Delta_{11}^2 I_1 \cdot I_1 + 2\, \Delta_{11}\Delta_{21} I_1 \cdot I_2 + 2\Delta_{11}\Delta_{31} I_1 \cdot I_3 + \Delta_{21}^2 I_2 \cdot I_2 +$$

$$2\Delta_{21}\Delta_{31} I_2 \cdot I_3 + \Delta_{31}^2 I_3 \cdot I_3$$

$$\bar{I}_{11} = \Delta_{11}^2 I_{11} + 2\Delta_{11}\Delta_{21} I_{12} + 2\Delta_{11}\Delta_{31} I_{13} + \Delta_{21}^2 I_{22} + 2\Delta_{21}\Delta_{31} I_{23} + \Delta_{31}^2 I_3$$

etc.

It is evident that these two pairs of transformations are isomorphic. It follows that symbolic multiplication using the notation of Clebsch Gordan gives us the fundamental solutions of (I).

For as **these** authors write

$$a_i \cdot a_j \equiv a_{ij} \equiv a_{ji}$$

we in our turn will write

$$I_i \cdot I_j \equiv I_{ij} \equiv I_{ji}$$

In $F_i \cdot F_j$ replace $I_i \cdot I_j$ by I_{ij}. Then write

$$F_i \cdot F_j \equiv F_{ij}$$

Clearly the F_{ij} are the solutions of (I). (See theorem page 34.) So we write

$$F_1 \cdot F_1 \equiv F_{11} \equiv \{\Theta^{\frac{1}{4}}(y_1 I_2 - y_2 I_1)\}^2 \equiv \Theta^{\frac{1}{2}}(y_1^2 I_{22} - 2y_1 y_2 I_{12} + y_2^2 I_{11})$$

$$F_1 \cdot F_2 \equiv F_{12} \equiv \Theta^{\frac{1}{4}}\{y_1 I_2 - y_2 I_1\{\}\ Y_1 I_2 - Y_2 I_1\}$$

$$\equiv \Theta^{\frac{1}{4}}\{y_1 Y_1 I_{22} - (y_1 Y_2 + y_2 Y_1) I_{12} + y_2 Y_2 I_{11}\}$$

$$F_2 \cdot F_2 \equiv F_{22} \equiv \{Y_1 I_2 - Y_2 I_1\}^2 \equiv Y_1^2 I_{22} - 2\, Y_1 Y_2 I_{12} + Y_2^2 I_{11}$$

$$F_1 \cdot F_3 \equiv F_{13} \equiv \Theta^{\frac{1}{4}}\{y_1 I_2 - y_2 I_1\} \cdot \Theta^{-\frac{1}{4}} D\, I_3 \equiv D\{y_1 I_{23} - y_2 I_{13}\}$$

$$F_2 \cdot F_3 \equiv F_{23} \equiv \{Y_1 I_2 - Y_2 I_1\} \cdot \Theta^{-\frac{1}{4}} D\, I_3 \equiv \Theta^{-\frac{1}{4}} D\, \{Y_1 I_{23} - Y_2 I_{13}\}$$

$$F_3 \cdot F_3 \equiv F_{33} \equiv \Theta^{-\frac{1}{4}} D\, I_3 \cdot \Theta^{-\frac{1}{4}} D\, I_3 \equiv \Theta^{-\frac{1}{2}} D^2 I_{33}$$

We have now obtained all the fundamental solutions of (I) by symbolic multiplication. The author has proved that these solutions are correct by *direct calculation*.

We are now in a position to give the result at which we have been aiming.

The integration of system (I) and (II) is equivalent to the integration of two systems of Riquier.[1] Consequently, with the aid of our solutions of (I) we easily compute a complete system of tensors for the case $r = 2$.

The general solution of (B) is written

$$\Phi(F_{11}, F_{12}, F_{13}, F_{22}, F_{23}, F_{33}; C_r)$$

---

1. See Requier, *Les Systèmes D'Équations Aux Dérivées Partielles*, Chapter XIII page 502.

where Φ is an arbitrary function and the C_r are the fundamental covariants of system (S) which Wilczynski has already calculated. But the system (A) given above represents the first infinitesimal transformations δI_{ij} of the tensors I_{ij}. (See page 9). Hence our tensors are the solutions of (A). To solve (A) we proceed thus.[1]

Let the Φ_{ij} be six independent functions of F_{ij} and C_r. Write

$$\Phi_{11} = \Phi_{12} = \Phi_{13} = \Phi_{22} = \Phi_{23} = \Phi_{33} = 0$$

Solve with respect to F_{11}, F_{12}, ... F_{33}. We get

$$(E) \ldots\ldots F_{ij} = \Psi_{ij}(C_r) \qquad (i,j = 1,2,3) \quad j \geqq i$$

where the Ψ are arbitrary functions. Write

$$M \equiv \frac{D(F_{11}, F_{12}, \ldots\ldots F_{33})}{D(I_{11}, I_{12}, \ldots\ldots I_{33})}$$

The F_{ij} are linear with respect to the I therefore M does not depend on the I.

Now we solve (E) *with respect to the* I_{ij} *and get*

$$I_{ij} = \frac{1}{M} \sum_{1}^{3}{}^{k} \sum_{1}^{3}{}^{l} M_{ij}^{kl} \Psi_{kl} \qquad j \geqq i \qquad l \geqq k$$

where M_{ij}^{kl} are the minors of M

The above formulae give us a functionally complete system of tensors associated with system (S) *when* $r = 2$.

In *a similar fashion one can compute a complete system of tensors when* $r = 1$.

1. The process is described by Riquier in Chapter XIII of his book.

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SOME OPERATIONAL METHODS IN THE CALCULUS OF FINITE DIFFERENCES

Joseph Talacko

1. Introduction.

Many formulas of the calculus of finite differences are very complicated. They can be simplified somewhat by introducing operational symbols for numerical interpolation, numerical differentiation and integration, for summation and solutions of difference equations. Special forms may be derived readily by the use of operators in some operational schemes. Whenever we have equidistant intervals, the application of these operators enables one to memorize complicated formulas in several basic relations, especially for central or mean differences.

In this paper a general operator, which possesses the distributive, commutative and associative properties is introduced as a generalization of calculus of symbols [1], [3], [8]. The standard operators of calculus of finite differences are special cases. Furthermore, this general operator may be expanded in infinite series and the inverse operator defined. The introduction of hyperbolic functions and their polynomial expansions are also demonstrated in this paper.

2. The Calculus of Symbols.

Let $f(x)$ be a real, single-valued, continuous function in a closed interval $a \leq x \leq b$, possessing a continuous differential coefficient of order k. Let $h_1, h_2, h_3, \ldots, h_n; H_1, H_2, \ldots, H_n$, be real constants .

Definition: An operator H is defined by the relation

$$(2.1)\quad Hf(x) \equiv H_1 f(x + h_1) + H_2 f(x + h_2) + \ldots + H_n f(x + h_n).$$

From (2.1) we get some particular cases:

If $h_1 = 0$, $H_i = 0$, $i = 2, 3, \ldots, n$; we get

$$H = H_1.$$

A special operator $H = 1$, does not change the given function and a zero operator is defined to be $H = 0$. Two H operators are equal if, and only if, their application to any function gives identical results. Let

$$(2.2) \quad \begin{aligned} K f(x) &= K_1 f(x+k_1) + \dots + K_r f(x+k_r), \\ Lf(x) &= L_1 f(x+l_1) + \dots + L_s f(x+l_s). \end{aligned}$$

The necessary and sufficient condition for $K = L$ is that

$$r = s, \quad K_i = L_i, \quad k_i = l_i \qquad (i = 1, 2, 3, \dots, s).$$

This operator possesses the distributive, commutative and associative properties. It permits the formation of polynomials and combination of these polynomials. From the definition (2.1) we get

$$(2.3) \qquad H\{f(x) + g(x)\} = Hf(x) + Hg(x)$$

Furthermore:

$$(2.4) \qquad (K + L)f(x) = Kf(x) + Lf(x)$$

$$(2.5) \qquad \begin{cases} K[Lf(x)] = [KL]f(x), \\ L[Kf(x)] = [LK]f(x), \end{cases}$$

and simply

$$KL = LK.$$

By induction, we may write,

$$H(KL) = (HK)L$$

$$(H + K)L = HL + KL$$

$$(H + K)L = L(H + K).$$

From the definition of the H-operator (2.1), we may write also an equation

$$(2.6) \qquad H + X = K$$

with an unknown operation X.

Let us define the operation of division by

$$(2.7) \qquad Hf(x) = g(x).$$

This operator H^{-1} possesses the distributive properties, but it is not always commutative.

Thus successive operations are indicated in reverse order. That is,

$$(2.8) \qquad HH^{-1}g(x) = g(x).$$

The operator HH^{-1} gives the unique result

(2.9) $$HH^{-1} = 1.$$

The H-operator may be generalized by introduction of the operator H_r, for which the limit is

(2.10) $$\lim_{r\to r_0} H_r f(x) = \lim_{r\to r_0} g(x,r) = g(x)$$

and simply

$$H = \lim_{r\to r_0} H_r .$$

Similarly, we define

(2.11) $$\varphi\,(H) = \lim\,(b_0 + b_1H + b_2H^2 + \ldots + b_n H^r),$$
$$= \lim_{r\to\infty} \sum_{i=0}^{r} b_i H^i$$

the symbolical infinite series. It converges, if the H-operator is the zero operator.

3. Special Operators in the Calculus of Finite Differences.

Let tabulated values of the function $f(x)$ be

$$f(x_0), \quad f(x_1), \quad f(x_2), \ \ldots$$

and let the intervals be equidistant:

$$x_i - x_{i-1} = x_{i+1} - x_i = h.$$

For the operator H, we have

(3.1) $$H f(x) = f(x + h)$$

which is equal to the well known operator E. For every integer n, define an exponential operator

$$H^n f(x) = f(x + nh).$$

For every integer m

$$Kf(x) = f(x + \frac{n}{m}\, h).$$

We have the relations:

$$K^m = H^n, \quad K = H^{n/m}.$$

Of course, instead of a rational number n/m, we may have any real number μ and the sequence

$$\mu_1, \mu_2, \cdots, \mu_r, \cdots,$$

if we write

$$\lim_{r\to\infty} \mu_r = \mu,$$

we may write

$$\lim_{r\to\infty} H^{\mu r} f(x) = \lim_{r\to\infty} f(x + \mu_r h) = f(x + \mu h).$$

Because $f(w)$ is continuous in the neighborhood of $w = x + \mu h$, we may write simply

$$\lim_{r\to\infty} H^{\mu r} = H^{\mu}.$$

Special operators, with respect to the interval h are:

a. The displacement operator E :

$$E f(x) = f(x + h). \tag{3.3}$$

b. The forward difference operator Δ :

$$\begin{gathered} \Delta = E^h - 1 \\ \Delta f(x) = f(x + h) - f(x). \end{gathered} \tag{3.4}$$

c. The backward operator ∇ :

$$\nabla = E^{-h} - 1 \tag{3.5}$$

where

$$\nabla f(x) = f(x + h) - f(x) = -\nabla f(x - h).$$

d. The central-difference operator δ :

$$\delta = E^{h/2} - E^{-h/2} \tag{3.6}$$

with

$$\delta = f(x + \frac{h}{2}) - f(x - \frac{h}{2}).$$

e. The mean-difference operator

$$\begin{gathered} \mu = \tfrac{1}{2} (E^{h/2} + E^{-h/2}) \\ \mu f(x) = \tfrac{1}{2} \{f(x + \frac{h}{2}) + f(x - \frac{h}{2})\}. \end{gathered} \tag{3.7}$$

f. The differential operator D :

$$D = h \cdot \lim_{a\to 0} \frac{E^a - 1}{a}. \tag{3.8}$$

This operator gives the differential of the function $f(x)$ with increment h.

$$D f(x)= h \cdot \lim_{a \to 0} \frac{E^a - 1}{a} = h \cdot \lim_{a \to 0} \frac{f(x + a) - f(x)}{a} = hf'(x).$$

Recalling the definitions of the displacement and other difference operators, we may write recurrence formulas relating differences and differentials and find some simple formulas of interpolation, numerical differentiation, integration and summation.

We may get other usable operators, by the introduction of hyperbolic function. Let us first introduce the familiar operational identity.

$$E = 1 + \Delta = e^D \tag{3.9}$$

and the fundamental identities involving central differences, hyperbolic sine and hyperbolic cosine:

$$\begin{cases} \delta = E^{h/2} - E^{-h/2} = e^{D/2} - e^{-D/2} \\ \delta = 2 \sinh (D/2) \end{cases} \tag{3.10}$$

$$\begin{aligned} \mu &= \tfrac{1}{2}(E^{h/2} + E^{-h/2}) = \tfrac{1}{2}(e^{D/2} + e^{-D/2}) \\ \mu &= \cosh (D/2). \end{aligned} \tag{3.11}$$

Solving for the operator D in (3.9) we get

$$D = ln(1 + \Delta) = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{\Delta^r}{r} . \tag{3.12}$$

This operator D may be written in the form

$$D = 2 \sinh^{-1}(\frac{\delta}{2}) = 2 \; ln \; (\frac{\delta}{2} + \sqrt{1 + \delta^2/4}), \tag{3.13}$$

or after substitution

$$D = 2 \cosh^{-1}\mu = \sqrt{1 + \delta^2/4} . \tag{3.14}$$

Many other relations may be found and various algebraic artificies for manipulating the basic identities may be discussed. We will show the usefulness of some operators which are similar to those of Lehmer [4], [5], and Michel [6].

4. *Some Formulas for Numerical Differentiation*

First, it is possible, taking powers of (3.12) step by step to get the well known formula of numerical differentiation of tabulated functions $f(x)$, based on (3.9) in the form:

$$D^{(n)} = \left[\sum_{r=1}^{\infty} (-1)^{r-1}\frac{\Delta^r}{r}\right]^n \tag{4.1}$$

The central difference formula offers more elegant results from (3.13) for $n = 2m$

$$\begin{aligned} D^{(2m)} &= [2\sinh^{-1}(\delta/2)]^{2m} \\ &= \delta^{2m}[1 - \frac{1}{2}\cdot\frac{1}{3}(\delta/2)^2 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5}(\delta/2)^4 - \dots]^{2m}. \end{aligned} \tag{4.2}$$

As a particular case we have for $m = 2$.

$$D^4 = \delta^{-4}[1 - \frac{1}{6}\delta^2 + \frac{7}{240}\delta^4 - \dots]^4.$$

If $n = 2m + 1$, we can write

$$\begin{aligned} D^{(2m+1)} &= [2\sinh^{-1}(\delta/2)]^{2m+1} \\ &= \mu\delta^{2m+1}[1 - \frac{1}{2}\cdot\frac{1}{3}(\delta/2)^2 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5}(\delta/2)^4 - \dots]^{2m+1}. \\ &\quad [1 - \frac{1}{2}(\delta/2) + \frac{1}{2}\cdot\frac{3}{4}(\delta/2)^4 - \dots]. \end{aligned} \tag{4.3}$$

Consider the differentiation with respect to δ of the equation (4.2) using the order as power exponent

$$\frac{d}{ds}D^{2m} = 2m\,[2\sinh^{-1}(\delta/2)]^{2m-1}\cdot(1 + \delta^2/4)^{-\frac{1}{2}} = \frac{2m}{\mu}D^{2m-1}, \tag{4.4}$$

hence

$$D^{2m-1} = \frac{\mu}{2m}\cdot\frac{d}{d\delta}D^{2m}.$$

In particular, we get for example

$$D^3 = \mu[\delta^3 - 1/4\,\delta^5 + 7/120\,\delta^7 - \dots].$$

5. *Some Interpolation Formulas.*

Using (3.1) we may get the Newton forward interpolation formula, for an interval of length h:

(5.1) $$E^{hn} = (1+\Delta)^n = 1 + \binom{n}{1}\Delta^2 + \binom{n}{2}\Delta + \dots + \binom{n}{r}\Delta^r + \dots$$
$$= 1 + \sum_{r=1}^{\infty}\binom{n}{r}\Delta^r.$$

and similarly the Newton backward formula becomes

(5.2) $$E^{-hn} = (1+\nabla)^n = 1 + \sum_{r=1}^{\infty}\binom{n}{r}\nabla^r.$$

The Stirling interpolation formula, using relations (3.6), (3.10) and (3.11) may be written for $n = 2m$

(5.3) $$E^{nh/2} = \cosh 2m\frac{D}{2} + \sinh 2m\frac{D}{2}.$$

The expansion of sinh mD and cosh mD is simple in terms of sinh D. Let

$$\sinh D = t \quad \text{so that} \quad D = \sinh^{-1} t \quad \text{and}$$

$$\sinh mD = \sinh(m \cdot \sinh^{-1} t) = f(t).$$

Differentiating twice with respect to t, if $f(0) = 0$, and $f'(0) = m$ we get the differential equation

(5.4) $$(1+t^2)f''(t) + tf'(t) - m^2 f(t) = 0$$

Using the Leibnitz theorem, differentiating r-times, for $t = 0$, we get a recurrence relation

(5.5) $$f^{(r+2)}(0) = (m^2 - r^2)f^{(r)}(0).$$

Equations (5.4) and (5.5), with initial values of $f(0)$, $f'(0)$ enable us to obtain the successive coefficients in the Maclaurin series of sinh (mD). This gives us the Stirling formula:

$$E^{mh} = m\sinh D + \frac{m(m^2-1)}{3\,!}\sinh^3 D + \frac{m(m^2-1)(m^2-3^2)}{5/!}\sinh^5 D + \dots$$

Expressing sinh D in terms of half arguments, we obtain

$$E^{mh} = 2m\sinh(D/2)\cosh(D/2) + \frac{2m(4m^2-4)}{5\,!}\sinh^3(D/2)\cosh(D/2) +$$

$$+ \frac{2m(4m^2-4)(4m-4)}{5!} \sinh^5(D/2)\cosh(D/2) + \dots +$$

$$+ \quad 1 + \frac{4m^2}{2!} \sinh(D/2) + \dots$$

finally

$$E^{mh} = 1 + m\mu\delta + \frac{m^2}{2!}\delta^2 + \frac{m(m^2-1)}{3!}\mu\delta^3 + \frac{m^2(m^2-1)}{4!}\delta^4$$

$$(5.6) \qquad + \frac{m(m^2-1)(m^2-2^2)}{5!}\mu\delta^5 + \frac{m^2(m^2-1)(m^2-2^2)}{6!}\delta^6 + \dots$$

If we let $\varphi(t) = \cosh(mD) = \cosh(m \sinh^{-1} t)$ we get Bessel's interpolation formula from (5.4):

$$f(0),\ f'(0) = 0$$

$$E^{(m-\frac{1}{2})h} = E^{(2m-1)h/2}$$

$$= 1 + \frac{m^2}{2!}\sinh^2 D + \frac{m^2(m^2-2^2)}{4!}\sinh^4 D + \frac{m^2(m^2-2^2)(m^2-4^2)}{6!}\sinh^6 D + \dots$$

or

$$E^{2(m-\frac{1}{2})h} = \cosh(2m-1)\frac{D}{2} + \sinh(2m-1)\frac{D}{2}$$

$$= \cosh(D/2) + \frac{2m(2m-2)}{2!}\sinh^2(D/2)\cosh(D/2)$$

$$+ \frac{2m(2m-2)(2m-4)(2m+2)}{4!}\sinh^4(D/2)\cosh(D/2) + \dots$$

$$+ (2m-1)\sinh(D/2) + \frac{(2m-1)2m(2m-2)}{3!}\sinh^3(D/2)$$

$$+ \frac{(2m-1)(2m)(2m-2)(2m-4)(2m+2)}{5!}\sinh^5(D/2) + \dots$$

Finally

$$E^{(2m-1)h/2} = \mu + (m - \tfrac{1}{2})\delta + \frac{m(m-1)}{2!}\mu\delta^2$$

$$(5.7) \qquad + \frac{m(m-1)(m-\frac{1}{2})}{3!}\delta^3 + \frac{m(m-1)(m+1)(m-2)}{4!}\mu\delta^4$$

$$+ \frac{m(m-1)(m+1)(m-2)(m-\frac{1}{2})}{5!}\delta^5 + \dots$$

6. *Inverse Operators* Δ^{-1}; D^{-1}, $\mu\,\delta^{-1}$.

The meaning of the inverse operator, as the operator of integration is best demonstrated by the equation

$$f(x) = D^{-1}g(x) = \frac{1}{h}\int g(x)\,dx + C, \tag{6.1}$$

which is the inverse operation of differentiation

$$Df(x) = g(x) = hf'(x)\ .$$

The indefinite integral D^{-1} leaves an arbitrary constant of integration. So, as we find the operational expressions for numerical differentiation, we may find operational formulas for numerical integrations and summations. If the operators are taken between definite limits, the constant of integration vanishes, and we may write

$$[D^{-1}g(x)]_a^b = \int_a^b g(x)\,dx.$$

To find Newton's integration formula, we use the following expansion of the operator D; as a new operator $(e^D - 1)^{-1}$:

$$(e^D - 1)^{-1} = D^{-1} + \frac{B_2}{2!}D - \frac{B_4}{4!}D^3 + \frac{B_6}{6!}D^5 - \ldots ,$$

where B_{2r} as known Bernouli numbers [4], [5]. The operator $(e^D - 1)^{-1}$ satisfies the equation

$$\Delta\ (D^{-1} + \frac{B_2}{2!}D - \frac{B_4}{4!}D^3 + \ldots) = (e^D - 1)(D^{-1} + \frac{B_2}{2!}D - \frac{B_4}{4!}D^3 + \ldots)\ .$$

It is a reciprocal operator to Δ. It is not the most general operator, but all results of the operation (6.2) will be a subset of results of the operation Δ^{-1}. We may write

$$\Delta^{-1} = D^{-1} + \frac{B_2}{2!}D - \frac{B_4}{4!}D^3 + \frac{B_6}{6!}D^5 - \ldots \tag{6.3}$$

or

$$D^{-1}f(x) = 1/h\ \int f(x)\,dx$$

$$= \Delta^{-1}f(x) - \frac{B_2}{2!}Df(x) + \frac{B_4}{4!}D^3f(x) - \frac{B_6}{6!}D^5f(x) + \ldots$$

which is the known Euler-Maclaurin formula. In general we get

$$D^{-1} = (\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \dots)^{-1}$$
$$= \Delta^{-1} + \frac{1}{2} - \frac{1}{12}\Delta + \frac{1}{24}\Delta^2 - \frac{19}{720}\Delta^3 + \dots$$

and finally, the Newton integration formula

(6.5) $$\frac{1}{h} \quad f(x)\,dx = \Delta^{-1}f(x) + \frac{1}{2}f(x) - \frac{1}{12}\Delta f(x)$$
$$+ \frac{1}{12}\Delta^2 f(x) - \frac{19}{790}\Delta^3 f(x) + \dots$$

Similarly, the operator δ^{-1}, represents the inverse operator, in the sense that it nullifies the other operation:

(6.6) $$\mu\delta^{-1}f(x) = ½f(x) + f(x-1) + f(x-2) + \dots + C$$

and for definite limits

$$[\mu\delta^{-1}]_a^b = ½f(b) + f(b-1) + \dots + f(a+1) + ½f(a).$$

The summation formulas may be developed by applying the procedure of differentiating with respect to an operator, finding the new operator, say $\psi(x)$, and taking the limits a, b.

7. *Other Extensions of Calculus of Symbolic Operators.*

By the same method as we have empolyed in sections 2,3 and 6, it is possible to define interpolation formulas in one exponential operator, by a method similar to the way Lehmer [6] defined the Bernoulli numbers. For example, the discussed Bessel's formula may be defined as a primitive function, using hyperbolic function

$$e^{(m-\frac{1}{2})D} = \frac{\cosh(m - ½)D}{\cosh(D/2)}\mu + \sinh(m - ½)D.$$

If we consider the other operator, say $(E^m - C)^n$, [6], [8], we may develop a simple procedure for solutions of difference equations. If we consider the infinite series in some operators, through functional expressions, we get a reliable tool with may applications.

* * * * * * * * * *

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MISCELLANEOUS NOTES

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A FINITE SEQUENCE AND A CARD TRICK

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Suppose we take a number of cards, or we might choose, for example, C blank cards and write a number on each of them with repitition allowed. We pick a number a, larger than all of the numbers written on the cards. Holding the cards face up, we count from whatever number is on the top card, up to a. In this way we obtain a pile of cards. We put this pile down in such a way that the card from which we started counting is on the top of the pile, but not necessarily face up. In fact it is better to have it face down. We continue this counting until either we exhaust the original pack of cards, or until we get a remainder r such that $r < a$. Without turning up the top cards we can find N, the sum of the numbers on those cards, knowing only C, the number of the cards, p, the number of the piles, r, the remainder, and a.

Let n_1, n_2 , ..., n_p be the numbers written on the top cards of the first, second, ..., p th. pile respectively, and N_1 , N_2, ..., N_p be the numbers of cards in the corresponding piles. It is easily observed that

$$N_1 = a - n_1 + 1,$$
$$N_2 = a - n_2 + 1,$$
$$\cdots \quad \cdots \quad \cdots$$
$$N_p = a - n_p + 1.$$

Adding these equalities we get

$$N_1 + N_2 + \ldots + N_p = pa - (n_1 + n_2 + \ldots + n_p) + p.$$

But $\quad n_1 + n_2 + \ldots n_p = N, \quad$ and $\quad N_1 + N_2 + \ldots + N_p = C - r.$

Therefore $\quad N = p(a + 1) + r - C.$

Now we have the formula. Let us entertain our friends. Pick up a deck of 52 cards and let $J = 11$, $Q = 12$, and $K=13$. Suppose $a=15$.

Ask a friend to do the counting in the way described above in the general case. As an example, he may see 3 on the top and he counts 3, 4, ..., 15, paying no attention to the numbers on the cards after the 3. Let the next card on the top be 6. Then the counting will be 6, 7, ..., 15. Next he sees 2 on the top and he counts 2, 3, ..., 15. Let the next card be $J = 11$. He counts 11, 12, ..., 15. Now a King comes on the top and he says 13, 14, 15. Finally the card on the top is 7, he counts 7, 8, ..., 13 and there are no more cards. Of course you are not watching him count the cards, in fact, you may be in a different room. He only tells you that he has 5 piles and 7 cards are left over. Substituting in the formula for N we have

$$N = 5(16) + 5 - 52 = 35.$$

So you say that the sum of the top cards is 35. Clearly the top cards in this example are 3, 6, 2, 11, 13, whose sum is 35.

There is a short cut for getting N. We leave it to the reader to find it.

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(Miscellaneous Notes continued on page 41)

MULTIPLE NUMBERS

John A. Tierney and John Tyler

A complex number is a number of the form $x + iy$ where the components x and y are real and $i = \sqrt{-1}$. If we replace $x + iy$ by $x + ry$ where $r^2 = -1$ we may regard $x + ry$ as representing two complex numbers $x + iy$ and its conjugate $x - iy$. It is interesting to consider the generalization to n components where r satisfies an equation of degree n.

We define a multiple number as a number having the form $c = \sum_{i=0}^{n} x_i r^i$ where the x_i are real and r satisfies the equation $r^{n+1} =$

$\sum_{i=0}^{n} a_i r^i$, a_i real, with the latter equation having no multiple roots. c is a multiple number in that it represents simultaneously $n + 1$ complex numbers.

We consider the case $n = 1$ and take $c = x + ry$ where r satisfies $r^2 = pr + q$ having roots

$$(1) \qquad r_1,\ r_2 = \frac{p}{2} \pm \frac{\sqrt{p^2 + 4q}}{2} = \frac{p}{2} \pm k \qquad [k \neq 0.]$$

We call x and y the components of c and $x_1 + ry_1$ and $x_2 + ry_2$ are defined to be equal if $x_1 + r_1 y_1 = x_2 + r_1 y_2$ and $x_1 + r_2 y_1 = x_2 + r_2 y_2$. If $x_1 + ry_1 = x_2 + ry_2$ their components are equal, otherwise the equation $(x_1 - x_2) + (y_1 - y_2)r = 0$ would have two distinct roots.

The norm N of c is the product of $x + r_1 y$ and its conjugate $x + r_2 y$ or $N(c) = x^2 + (r_1 + r_2)xy + r_1 r_2 y^2 = x^2 + pxy - qy^2$. The conjugate of $x + ry$ is $x + (p - r)y$ since their product is N. To obtain a polar form for c, let $c = x + ry = \Psi e^{r\theta}$

Then

$$[c(r_1)][c(r_2)] = N = \psi^2 e^{(r_1 + r_2)\theta} = \psi^2 e^{p\theta}$$

and

$$\psi = \sqrt{N}\, e^{-p/2\ \theta}$$

Thus

$$c = x + ry = \sqrt{N}\, e^{(r-p/2)\theta}$$

We call $\sqrt{N}$ the modulus of c and θ the amplitude of c. The multiple number $e^{(r-p/2)\theta}$ has unit modulus and we write

$$e^{(r-p/2)\theta} = c(\theta) + rs(\theta). \tag{2}$$

To find the functions c and s we substitute r_1 and r_2 in (2) and solve for c and s to obtain

$$c(\theta) = \cosh k\theta - \frac{p}{2k} \sinh k\theta \tag{3}$$

and

$$s(\theta) = \frac{1}{k} \sinh k\theta \tag{4}$$

From $x + ry = \sqrt{N}\,[c + rs]$ we find that $x = \sqrt{N}\,c$; $y = \sqrt{N}\,s$ and

$$\theta = \frac{1}{k} \sinh^{-1} \frac{ky}{\sqrt{N}} \tag{5}$$

From (2) we also have $[c(\theta) + rs(\theta)]^n = c(n\theta) + rs(n\theta)$, a generalization of De Moivre's theorem. For $p = 0$, $q = 1$ this reduces to $[\cosh\theta + r\sinh\theta]^n = \cosh n\theta + r\sinh n\theta$ and we can obtain $\cosh n\theta$ or $\sinh n\theta$ in terms of functions of $\cosh\theta$ and $\sinh\theta$ by expanding and equating components.

To obtain the relationship between $c(\theta)$ and $s(\theta)$ we substitute r_1 and r_2 in (2) and multiply the results.

$$1 = c^2 + (r_1 + r_2)cs + r_1 r_2 s^2$$

$$1 = c^2 + pcs - qs^2.$$

To find the addition formulas for the functions $c(\theta)$ and $s(\theta)$ we substitute θ_1 and θ_2 in (2) and multiply.

$$e^{(r-p/2)\theta_1}\, e^{(r-p/2)\theta_2} = e^{(r-p/2)(\theta_1+\theta_2)}$$

$$[c(\theta_1) + rs(\theta_1)]\,[c(\theta_2) + rs(\theta_2)] = c(\theta_1 + \theta_2) + rs(\theta_1 + \theta_2)$$

$$= c(\theta_1)\,c(\theta_2) + r[c(\theta_2)\,s(\theta_1) + c(\theta_1)\,s(\theta_2)] + (pr + q)\,s(\theta_1)s(\theta_2).$$

Equating components we obtain

$$c(\theta_1 + \theta_2) = c(\theta_1)\,c(\theta_2) + qs(\theta_1)\,s(\theta_2) \tag{6}$$

$$s(\theta_1 + \theta_2) = s(\theta_1)\,c(\theta_2) + s(\theta_2)\,c(\theta_1) + ps(\theta_1)\,s(\theta_2) \tag{7}$$

If in (6) and (7) we set $\theta_2 = -\theta_1 = -\theta$ we obtain

$$1 = c(\theta)c(-\theta) + qs(\theta)s(-\theta)$$

$$0 = s(\theta)c(-\theta) + s(-\theta)c(\theta) + ps(\theta)s(-\theta)$$

from which

$$s(-\theta) = \frac{-s(\theta)}{c^2(\theta) + pc(\theta)s(\theta) - qs^2(\theta)} = -s(\theta)$$

and

$$c(-\theta) = c(\theta) + ps(\theta).$$

Using these results in (6) and (7) we have

$$c(\theta_1 - \theta_2) = c(\theta_1)c(\theta_2) + pc(\theta_1)s(\theta_2) - qs(\theta_1)s(\theta_2)$$

$$s(\theta_1 - \theta_2) = s(\theta_1)c(\theta_2) - s(\theta_2)c(\theta_1).$$

Continuing, we are able to derive formulas analagous to the double and half-angle formulas. If we compute the modulus of the sum of two multiple numbers we obtain a generalized law of cosines. We are able to differentiate the functions $c(\theta)$ and $s(\theta)$ together with their inverses and finally we can show that the components of a function of a multiple number satisfy the partial differential equation $Uyy - qUxx - pUxy = 0$ which reduces to Laplace's eq. in one trig-case and the wave eq. in the hyperbolic case.

We illustrate one use of multiple numbers by solving the Diophantine equation

$$Z^3 = u^2 + v^2.$$

Let $c = x + ry$ where $r^2 = pr + q$. If $p = 0$, $q = -1$, $N(c) = x^2 + y^2$ and $c^3 = (x^3 - 3xy^2) + (3x^2y - y^3)r$.

Using $c = \psi e^{r\theta}$ and $N = \psi^2 e^{p\theta}$ it is easy to show that $[N(c)]^n = N(c^n)$, $n = 1, 2, 3, \ldots$. Hence for $n = 3$ we have

$$(x^2 + y^2)^3 = (x^3 - 3xy^2)^2 + (3x^2y - y^3)^2$$

from which

$$Z = x^2 + y^2$$

$$u = x^3 - 3xy^2$$

$$v = 3x^2y - y^3$$

This agrees with the general solution of elementary number theory.

U.S. Naval Academy

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C.N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions you would like others to discuss, should be sent to Joseph Seidling Alfred University, Alfred, New York.

Angle of Inclination and Curvature

David Gans

The familiar notion of angle of inclination of a straight line, as presented in texts on analytic geometry and assumed in calculus books, seems so natural and perfectly straightforward that it comes as something of a shock when we first learn that the notion involves a discontinuity when applied to even the simplest of curves. If a straight line g meets the x-axis in a point P, the angle of inclination of g, we know, is the positive angle, less than 180°, whose initial ray extends from P along the x-axis in the positive direction and whose terminal ray extends along g in the direction in which y increases. The angle of inclination of a straight line perpendicular to the y-axis is taken either as 0° or 180°.

To exhibit the discontinuity referred to, let us consider a very simple curve, say, one which is represented in some interval by a single-valued function $y = f(x)$ possessing at least a first and second derivative. At a point where the tangent line to this curve is not horizontal it is clear that the angle of inclination of this line is defined, in accordance with the above definition, and, moreover, is continuous in the neighborhood of the point. The angle of inclination of a horizontal tangent line at a point of inflection is not defined, the above definition being ambiguous for horizontal lines. Let us, then, agree to define it to be 0° at a point of this type on each side of which $f(x)$ is an increasing function, and to be 180° at a point of this type on each side of which $f(x)$ is a decreasing function. This agreement is reasonable since it achieves continuity for the angle of inclination at these points.

There remain the horizontal tangents at points of relative maximum or minimum. If we proceed along the curve and approach a point of relative maximum from the left, we find that the angle of inclination tends to 0°, whereas if we approach it from the right this angle tends to 180°. For a point of relative minimum the angle tends to 180° or to 0° according as the point is approached from the left or the right. It follows that, no matter which of the values 0°, 180° is taken as the angle of inclination at a point of relative maximum or minimum, there will be discontinuity at the point as far as the angle is concerned.

To sum up, using symbols, if α represents the variable angle of inclination of the tangent to the given curve, then α is a function of x: $\alpha = \alpha(x)$, which is defined (or definable) for all values of x in the given interval, and is continuous for all these values except those corresponding to relative maxima and minima of $f(x)$.

The function $\alpha = \alpha(x)$, of course, has no derivative at its points of discontinuity. A very large number of books on calculus, strangely enough, have overlooked this fact in their discussion of curvature in a plane. Differing somewhat in their definitions of curvature, these books all end up by computing it in the same way, namely, by finding the rate of change of the angle of inclination with respect to the length of arc. In the process they differentiate $\alpha = \alpha(x)$, after writing it in the ambiguous form $\alpha = \tan^{-1}[f'(x)]$, then substitute the result in the equation $d\alpha/ds = (d\alpha/dx)(dx/ds)$, and obtain

$$(1) \qquad K = y''/(1 + y'^2)^{3/2}$$

as the expression for curvature. They then proceed to use this formula to find the curvature at the vertex of the parabola $x^2 = 4ay$, at the end of the minor axis of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, at the point $x = \pi/2$ of the curve $y = \sin x$, and so forth, points for which their derivation of formula (1) is invalid because of the discontinuity of $\alpha(x)$.

Interestingly enough, the curvatures thus obtained at points of relative maximum or minimum are actually correct since formula (1) holds for such points despite the flaw in its derivation. This can be seen by noting that there is an alternative way of defining an angle of inclination which involves no ambiguity or discontinuity and which validates the faulty derivations without changing any of the formal steps. According to this alternative method, the angle of inclination, τ, of a straight line with slope m is the *principal* value of $\tan^{-1}m$. Thus, the angle τ is positive acute, negative acute, or zero according as m is positive, negative, or zero. Since $\tau = \tan^{-1}u$ is a continuous and differentiable function of u when τ is restricted to principal values, and $f'(x)$ is assumed to be differentiable throughout the given interval, so will $\tau = \tan^{-1}[f'(x)]$ be continuous and have a derivative, $d\tau/dx = f''(x)/(1 + [f'(x)]^2)$, throughout the interval. Substitution of this in $d\tau/ds = (d\tau/dx)(dx/ds)$ gives formula (1).

Fortunately, discussions of curvature can be improved without taking the somewhat drastic step of changing the definition of an angle of inclination. All that is necessary is that the angle τ of the alternative definition be used as a measure of the direction of a curve without calling it an angle of inclination. No special name for it seems necessary. Incidentally, this use of τ would also represent a significant application of the principal values of the inverse trigonometric functions, such applications being quite scarce in introductory courses in calculus.

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NOTES ON CIRCULAR AND HYPERBOLIC FUNCTIONS

William S. McCulley

1. One of the aims of instruction in calculus is to prepare students for solving differential equations. An important class of differential equations consists of ordinary equations, of second and higher order, with constant coefficients, which describe oscillating and non-oscillating electrical and mechanical systems. The former are quite naturally described by oscillating functions, i.e., circular sines and cosines. The latter are also quite naturally described by non-oscillating functions, i.e., hyperbolic sines and cosines.

The usual syllabus for calculus courses places much more emphasis on developing and using the properties of the circular functions, relegating the hyperbolic functions to a coverage of two or three lessons. The closely parallel analytical properties of the two types of function render it feasible and instructive to treat them together. Further, various relations between circular and hyperbolic functions are interesting to explore in themselves.

2. As an aid to remembering many of the relations between circular functions the following diagram, seen originally in the preface to Wentworth & Smith's Trigonometry, published in the early 1920's, serves as a useful condensation of a large amount of information.

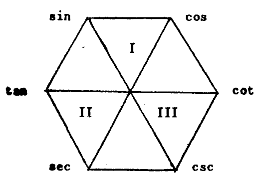

Figure 1.

It is easy to verify the following rules:

2.1 a. Functions and co-functions appear to left and right, respectively.

b. Derivatives of functions (on left) are positive. Derivatives of co-functions (on right) are negative.

2.2 Reciprocal functions appear at opposite ends of diagonals.

2.3 Each function equals the first function following divided by the second function following, in either direction.

E.g., $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$.

2.4 Each function equals the product of the two adjoining functions. E.g., cot $\theta = \cos\theta \csc\theta$.

2.5 The two functions at the top of the hexagon have finite amplitudes; the other four become infinite.

2.6 The Roman numerals indicate the pairing of functions in quadratic identities.

Similarity of definitions permits a similar, but not identical, arrangement of hyperbolic functions, exemplifying all the above rules except 2.1 a.

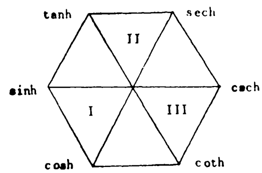

Figure 2.

We shall see in section 5 how a comparison of Figure 1 and Figure 2 yields further information about the graphs of circular and hyperbolic functions.

3. When we are given the value of one of the circular functions and wish to find the values of the other five functions without consulting tables, we may use the well-known device of reference triangles. A simple modification enables us to use a similar procedure for finding the values of hyperbolic functions (1). Suppose sinh $\theta = 3/5$. Draw two right triangles. On the first of these triangles assign values to the altitude and hypotenuse according to the definition of the circular sine. On the second triangle, assign the same pair of values, but rotated clockwise one place with respect to sides. Thus we have

Figure 3.

On the second triangle the length of the hypotenuse is obtained by means of the Pythagorean theorem. This value is assigned to the unnumbered side of the first triangle. The values of the remaining five hyperbolic functions may now be obtained from the numbers on the first triangle by applying the definitions of the corresponding circular functions.

4. The differentiation formulas for the circular and hyperbolic

functions and their inverses are assumed known. (See (2), (3), (4) for details.) It is pertinent to observe that teaching and learning these formulas by pairing or "stacking" them. E.g.,

$$D_x\binom{\cos}{\cosh}u) \quad J \binom{\sin}{\sinh} u\, D_x u, \quad J = -1 \text{ (circ.)}; \; J = +1 \text{ (hyp.)}$$

$$D_x(\text{arc} \binom{\tan}{\tanh} u) = \frac{D_x u}{1 \mp u^2}, \text{ upper function - upper sign, etc.}$$

5. Consider two parametrizations of the unit hyperbola $x^2 - y^2 = 1$:

$$(5.1) \quad \begin{array}{lll} x = \cosh \phi, & y = \sinh \phi, & \text{and} \\ x = \sec \theta, & y = \tan \theta. & \end{array}$$

We thus establish a correspondence between the hyperbolic angle ϕ and the circular angle θ expressed by

$$(5.2) \quad \cosh \phi = \sec \theta, \quad \sinh \phi = \tan \theta.$$

These relations can be symbolized by superimposing the hexagons of Figure 1 and Figure 2, from which alignment we can read off the remaining four equivalences:

$$(5.3) \quad \begin{array}{ll} \tanh \phi = \sin \theta, & \coth \phi = \csc \theta, \\ \operatorname{sech} \phi = \cos \theta, & \operatorname{csch} \phi = \cot \theta. \end{array}$$

If we solve each of these, in particular (5.2), for θ in terms of ϕ, or for ϕ in terms of θ, we get

$$(5.4) \quad \theta = \text{arc tan } (\sinh \phi), \; \phi = \text{arc sinh } (\tan \theta), \text{ etc.}$$

It is clear from these relations that the domain $(-\infty, \infty)$ for ϕ corresponds to the domain $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ for θ, and that $\phi = 0$ corresponds to $\theta = 0$. We see that for corresponding values on the two domains the functions in (5.2) and (5.3) have equal function values, where they are defined, or else they both become infinite. Comparison of the evaluated derivatives of the functions in (5.2) and (5.3) shows that the corresponding functions have equal slopes at corresponding points of their domains.

It can be verified by differentiation that $d\theta/d\phi = \operatorname{sech} \phi$ for each of the relations represented by (5.4a); differentiation of each of the six represented by (5.4b) leads to $d\phi/d\theta = \sec \theta$. That $d\theta/d\phi = (d\phi/d\theta)^{-1}$ may be seen from (5.2a). If we reverse these differentiations we obtain six equivalent integration formulas for sech ϕ and and for sec θ.

If we wish to derive such formulas by formal integration, we may may proceed as follows:

(5.5) $$\operatorname{sech}\phi\, d\phi = \int \frac{d\phi}{\cosh\phi} = \int \frac{\cosh\phi\, d\phi}{1+\sinh^2\phi} + \text{arc tan } (\sinh\phi) + C,$$

etc.

It is interesting to compare the number of operations necessary to evaluate antiderivatives of the form(5.4b)and(5.5)with the number required for evaluating the standard antiderivatives. In the case of (5.4b)the standard formula is $\int \sec\,\theta d\theta = \ln(\sec\theta + \tan\theta) + C$, which requires three references to tables and one addition;(5.4b) and (5.5) can be evaluated with two references, or, on a vector slide rule, by one setting of the indicator.

In his work published in 1830 Gudermann used the relation(5.5) to define the function he named the longitude. Cayley later renamed it the gudermannian. Thus

(5.6) $$\operatorname{gd}\phi = \int \operatorname{sech}\phi\, d\phi = \text{arc tan } (\sinh\phi) + C,$$

and its inverse

(5.7) $$\operatorname{gd}^{-1}\theta = \int \sec\theta\, d\theta = \text{arc sinh } (\tan\theta) + C.$$

From the discussion following(5.4)we find $\operatorname{gd}(0) = 0$, $\operatorname{gd}(\pm\infty) = \pm \tfrac{1}{2}\pi$.

6. From the correspondence relations $\cos\theta = \operatorname{sech}\phi$, $\sin\theta = \tanh\phi$, we may parametrize the unit circle $x^2 + y^2 = 1$ in terms of the hyperbolic angle ϕ by $x = \operatorname{sech}\phi$, $y = \tanh\phi$. Then the area of the portion of this circle in the first quadrant is given by the line integral

(6.1) $$A = \tfrac{1}{2}\int \begin{vmatrix} x & y \\ dx/d\phi & dy/d\phi \end{vmatrix} d\phi = \tfrac{1}{2}\int \begin{vmatrix} \operatorname{sech}\phi & \tanh\phi \\ -\operatorname{sech}\phi\tanh\phi & \operatorname{sech}^2\phi \end{vmatrix} d\phi$$

$$= \tfrac{1}{2}\int_0^\infty \operatorname{sech}\phi\, d\phi = \tfrac{1}{2}\operatorname{gd}\phi \Big|_0^\infty = \tfrac{1}{2}(\tfrac{1}{2}\pi - 0) = \frac{\pi}{4}$$

Similarly, the length of arc is given by

(6.2) $$s = \int_0^\infty (\operatorname{sech}^2\phi\,\tanh^2\phi + \operatorname{sech}^4\phi)^{\frac{1}{2}} d\phi = \int_0^\infty \operatorname{sech}\phi\, d\phi = \operatorname{gd}\phi \Big|_0^\infty = \tfrac{1}{2}\pi.$$

7. The functional correspondence established in (5.2)and(5.3) gives us another way of integrating $\sec^3\theta$, hence odd powers of sec 0, as follows:

(7.1) $$\begin{aligned} \int \sec\theta\, d\theta &= \int \cosh^3\phi\, \operatorname{sech}\phi\, d\phi = \int \cosh^2\phi\, d\phi \\ &= \tfrac{1}{2}(\sinh\phi\cosh\phi + \phi) + C \\ &= \tfrac{1}{2}(\tan\theta\sec\theta + \text{arc sihnh } (\tan\theta)) + C. \end{aligned}$$

The same functional correspondences may also be used to establish hyperbolic analogs of the Wallis formulas, thus

$$(7.2)\quad \begin{aligned} \int_0^{\frac{1}{2}\pi} \sin^n\theta\, d\theta &= \int_0^{\infty} \tanh^n\phi \operatorname{sech}\phi\, d\phi = \int_0^{\infty} \sinh^n\phi \cosh^{-n-1}\phi\, d\phi; \\ \int_0^{\frac{1}{2}\pi} \cos^n\theta\, d\theta &= \int_0^{\infty} \operatorname{sech}^{n+1}\phi\, d\phi = \int_0^{\infty} \cosh^{-n-1}\phi\, d\phi, \text{ etc.} \end{aligned}$$

8. As an example of the compact expression obtainable by combining circular and hyperbolic functions, consider the differential equation

$$(8.1)\qquad y_{tt} + ay_t + by = 0,$$

where a and b denote constants. The auxiliary equation is

$$r^2 + ar + b = 0,$$

of which the roots are

$$r = \frac{-a \pm (a^2 - 4b)^{\frac{1}{2}}}{2}.$$

The particular solution is given by

$$(8.1)\qquad y = e^{\alpha t}(A\overset{\cos}{\cosh}\,\beta t + B\overset{\sin}{\sinh}\,\beta t),$$

where $\alpha = -\frac{1}{2}a$, $\beta = \frac{1}{2}[J(a^2 - 4b)]^{\frac{1}{2}}$, $J = \dfrac{|a^2 - 4b|}{a^2 - 4b}$, with the rule:

If $J = +1$, use hyperbolic functions; if $J = -1$, use circular functions.

9. Additional analogies and generalizations involving circular and hyperbolic functions include the following:

a. The hyperbolic analog of de Moivre's theorem

$$(\cosh\phi + j\sinh\phi)^n = \cosh n\phi + j\sinh n\phi, \text{ where } j^2 = +1;$$

b. The generalization of circular and hyperbolic functions to oscillating and non-oscillating Bessel functions respectively.

c. The generalization of circular and hyperbolic functions to the elliptic functions. In particular, the Jacobian elliptic functions $sn(u|m)$, $cn(u|m)$ satisfy, for limiting values of the parameter m, the following equations (see [5])

$$\begin{aligned} sn(u|0) &= \sin u, & sn(u|1) &= \tanh u, \\ cn(u|0) &= \cos u, & cn(u|1) &= \operatorname{sech} u. \end{aligned}$$

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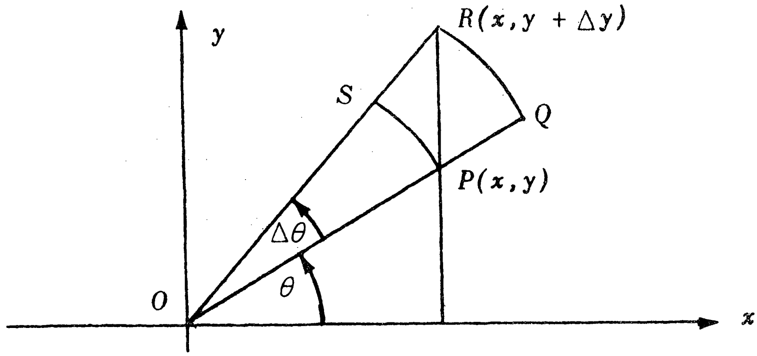
The Derivatives of the Trigonometric Functions

M. J. Pascual

The following method of arriving at the derivatives of the trigonometric functions might appeal to some, even if for the sole reason that it differs from the conventional method employing the $\lim\limits_{h\to 0} \frac{\sin h}{h}$. A novel feature of this derivation is the fact that the derivative of the $\sin\theta$ is arrived at last and then as a consequence we may obtain the above limit. The derivation will be for θ lying in the first quadrant and with $\Delta\theta$ positive. Obvious modifications can be made for other quadrants and $\Delta\theta$ negative.



We start with the $\tan\theta$ and $\cot\theta$. By definition

$$D(\tan\theta) = \lim_{\Delta\theta\to 0} \frac{\tan(\theta + \Delta\theta) - \tan\theta}{\Delta\theta}$$

$$= \lim_{\Delta\theta\to 0} \frac{\frac{y+\Delta y}{x} - \frac{y}{x}}{\Delta y} \cdot \frac{\Delta y}{\Delta\theta} = \frac{1}{x} \lim_{\Delta\theta\to 0} \frac{\Delta y}{\Delta\theta}$$

where $x \neq 0$, $\theta \neq \pi/2$. Similarly we obtain

$$D(\cot\theta) = -\frac{1}{y^2} \lim_{\Delta\theta\to 0} x \frac{\Delta y}{\Delta\theta}$$

where $y \neq 0$, $\theta \neq 0$. Now to obtain the $\lim\limits_{\Delta\theta\to 0} x \frac{\Delta y}{\Delta\theta}$, from the above diagram it is easy to see that

$$\text{area } OPS < \text{area } OPR < \text{area } OAR$$

$$\tfrac{1}{2}(x^2 + y^2)\,\Delta\theta < \tfrac{1}{2}x\,\Delta y < \tfrac{1}{2}[x^2 + (y + \Delta y)^2]\,\Delta\theta$$

hence

$$\lim_{\Delta\theta\to 0} x\frac{\Delta y}{\Delta\theta} = x^2 + y^2.$$

Therefore

$$D(\tan\theta) = \frac{x^2 + y^2}{x^2} \qquad \text{for} \quad x \neq 0,\ \theta \neq \pi/2,$$

$$D(\tan\theta) = \sec^2\theta$$

and

$$D(\cot\theta) = \frac{x^2 + y^2}{x^2} \qquad y \neq 0, \quad \theta \neq 0$$

$$D(\cot\theta) = -\csc^2\theta$$

By using the identities relating the trigonometric functions we easily get

(1) $D(\sec\theta) = \sec\theta\tan\theta$ and $D(\cos\theta) = -\sin\theta$ for $\theta \neq \pi/2$

(2) $D(\csc\theta) = -\csc\theta\cot\theta$ and $D(\sin\theta) = \cos\theta$ for $\theta \neq 0$.

To verify that the formulas for the derivatives of the $\sin\theta$ and the $\cos\theta$ hold for these exceptional values of θ as well, we have by the quotient formula that

$$D(\tan\theta) = \frac{\cos\theta\, D(\sin\theta) - \sin\theta\, D(\cos\theta)}{\cos^2\theta} \qquad \text{for } \theta \neq \pi/2$$

$$\sec^2\theta = \frac{\cos\theta\, D(\sin\theta) + \sin^2\theta}{\cos^2\theta}$$

solving for $D(\sin\theta)$ we get $D(\sin\theta) = \cos\theta$ for $\theta \neq \pi/2$. Combining this with (2) we may say that for $0 \leq \theta \leq \pi/2$

$$D(\sin\theta) = \cos\theta$$

Similarly by using $D(\cot\theta)$ for $\theta \neq 0$ we obtain $D(\cos\theta) = -\sin\theta$ for $\theta \neq 0$ which combined with (1) yields

$$D(\cos\theta) = -\sin \qquad \text{for } 0 \leq \theta \leq \pi/2.$$

Finally to evaluate the $\lim_{h\to 0}\frac{\sin h}{h}$ we have $D(\sin\theta) = \lim_{h\to 0}\frac{\sin(\theta+h) - \sin\theta}{h}$

which for $\theta = 0$ gives us $\cos 0 = \lim_{h\to 0}\frac{\sin h}{h}$ or $\lim_{h\to 0}\frac{\sin h}{h} = 1$.

Siena College, New York

ON A CHARACTERIZATION OF ORTHOGONALITY

Waleed A. Al-Salam

D. Dickinson [1] gave the following theorem:

Theorem: For a set of polynomials $p_n(x)$, $n = 0,1,2,\ldots$ where each p_n is of degree precisely n, to have the property

$$a_n = p_{n-1}(x)\, p_n(-x) + p_{n-1}(-x) p_n(x)$$

for some $a_n \neq 0$, $n = 1,2,\ldots$, it is necessary and sufficient that there exists a recurrence relation of the form

$$p_n(x) = xB_n\; p_{n-1}(x) + C_n p_{n-1}(x), \qquad C_n \neq 0 \quad (n = 2,3,\ldots)$$

and that $p_1(0) \neq 0$.

He also proved the following characterization of orthogonality

Corollary: For a set of polynomials $p_n(x)$, $n = 0,1,2,\ldots$ where each p_n is of degree n, to be orthogonal, it is sufficient that there exists a relation of the form

$$a_n = p_n(x)\; p_{n-1}(-x) + p_n(-x)\; p_{n-1}(x)$$

where $(-1)^n a_1 a_n < 0$, for $n \geq 2$.

In this note we obtain a more general criterion of orthogonality which actually reduces to Dickinson's criterion as a special case and further includes the classical polynomials. We also obtain the orthogonality of a class of polynomials associated with the orthogonal polynomials. We first prove the following theorem:

Theorem 1. Given two sequences of polynomials $\{f_n(x)\}$ and $\{g_n(x)\}$ $n = 0,1,2,\ldots$, where $f_n(x)$ and $g_n(x)$ are of degree n and $n-1$ respectively, $f_0 \neq 0$, $g_1(x) \neq 0$, $g_0 = 0$, then the necessary and sufficient condition that there exists a relation

$$(1) \qquad f_n(x)\; g_{n+1}(x) - f_{n+1}(x)\; g_n(x) = a_n \neq 0$$

is that $\{f_n\}$ and $\{g_n\}$ satisfy a recurrence relation of type

$$(2) \quad u_{n+1}(x) = (A_n x + B_n)\; u_n(x) + C_n u_{n-1}(x), \quad (C_n \neq 0, \quad n = 1,2,\ldots)$$

Proof: If $\{f_n(x)\}$ and $\{g_n(x)\}$ satisfy (2) then it is easily seen that

$$f_n(x)g_{n+1}(x) - f_{n+1}(x)g_n(x) = -C_n(f_{n-1}(x)g_n(x) - f_n(x)g_{n-1}(x))$$
$$= (-1)^n C_n C_{n-1} \cdots C_1 f_0 g_1$$
$$\neq 0 \text{ for all } n = 1, 2, \ldots$$

Now assume that (1) holds. Then by Casorati's theorem $\{f_n\}$ and $\{g_n\}$ are independent solutions of the difference equation

$$\begin{vmatrix} u_{n-1}(x) & f_{n-1}(x) & g_{n-1}(x) \\ u_n(x) & f_n(x) & g_n(x) \\ u_{n+1}(x) & f_{n+1}(x) & g_{n+1}(x) \end{vmatrix} = 0$$

By expanding we find

$$u_{n+1}(x) = \frac{f_{n-1}g_{n+1} - f_{n+1}g_{n-1}}{a_{n-1}} u_n(x) - \frac{a_n}{a_{n-1}} u_{n-1}(x).$$

Now consider $F_n(x) = f_{n-1}g_{n+1} - f_{n+1}g_{n-1}$. If we eliminate $g_{n+1}(x)$ by means of (1) we get

$$F_n(x) = \frac{f_{n-1}a_n + g_n f_{n+1} f_{n-1} - f_n f_{n+1} g_{n-1}}{f_n}$$
$$= \frac{f_{n-1}a_n + f_{n+1}a_{n-1}}{f_n}$$

But a_n does not involve x, and f_{n+1} is of degree $n + 1$, then we get that F_n must be of the first degree. Hence

$$u_{n+1}(x) = (A_n x + B_n)\, u_n(x) - \frac{a_n}{a_{n-1}} u_{n-1}(x)$$

This completes the proof of the theorem.

In particular if we have $g_n(x) = f_n(x) + (-1)^{n+1} f_n(-x)$ then the expression in the L.H.S. of (1) reduces to

$$(-1)^n \left(f_n(x)f_{n+1}(-x) + f_n(-x)f_{n+1}(x)\right)$$

and similarly the expression F_n reduces to

$$(-1)^n(f_{n-1}(x)f_{n+1}(-x) - f_{n+1}(x)f_{n+1}(-x))$$

from which we see that $F_n(0) = 0$ and hence $B_n = 0$. This shows that given a sequence $\{f_n(x)\}$ where f_n is a polynomial of degree $n, f_0 \neq 0$, then the necessary and sufficient condition that there is a relation

$$f_n(x)f_{n+1}(-x) + f_n(-x)f_{n+1}(x) = a_n \neq 0$$

is that $\{f_n\}$ satisfy a difference equation of type

$$u_{n+1}(x) = xA_n u_n(x) + C_n u_{n-1}(x), \quad C_n \neq 0 \tag{3}$$

which is Dickinson's theorem. Another way of stating this is that the necessary and sufficient consition for $\{f_n(x)\}$ and $\{(-1)^n f_n(-x)\}$ to be two independent solutions of (3) is that there is a relation

$$f_n(x)f_{n+1}(-x) + f_n(-x)f_{n+1}(x) = a_n \neq 0$$

for some a_n.

Now the following characterization of orthogonality can be proved by using Favard's theorem [2] and the previous theorems.

Theorem 2: Given a sequence $\{f_n(x)\}$ where f_n is a polynomial of degree n and $f_0 \neq 0$ then the necessary and sufficient condition that $\{f_n(x)\}$ be orthogonal is that there exists another sequence of polynomials $\{g_n(x)\}$ where g_n is a polynomial of degree $n-1$, $g_0 = 0$, $g_1(x) \neq 0$ such that

$$f_n(x)g_{n+1}(x) - f_{n+1}(x)g_n(x) = a_n \neq 0 \quad \text{and } a_0 a_n \neq 0.$$

It is well known that if $\{f_n(x)\}$ satisfy the recurrence relation (2) then we have for each $p = 1, 2, 3, \ldots$

$$f_{n+p}(x) = T_p^{(n)}(x)f_n(x) + S_p^{(n)}(x)f_{n-1}(x)$$

where $T_p^{(n)}(x)$ and $S_p^{(n)}(x)$ are polynomials in x of degrees p and $p-1$ respectively and where $T_0^{(n)}(x) = 1$, $T_1^{(n)}(x) = A_n x + B_n$, $S_0^{(n)} = 0$, and $S_1^{(n)}(x) = C_n \neq 0$. Let us call $T_p^{(n)}(x)$ and $S_p^{(n)}(x)$ for each fixed n the Lommel polynomials of the first and second kinds associated with $f_n(x)$ respectively. Then we can prove the following

Theorem 3: Given a sequence of orthogonal polynomials $\{f_n(x)\}$ where $\deg f_n = n$, and $f_0 \neq 0$, then the Lommel polynomials associated with each fixed n are also orthogonal.

Proof: Since the $\{f_n(x)\}$ are orthogonal, they satisfy a difference equation of the second order.

$$u_{n+1}(x) = (A_n x + B_n)\, u_n(x) - C_n y_{n-1}(x), \qquad C_n > 0$$

and there exists another sequence $\{g_n(x)\}$ such that

$$f_n(x)\, g_{n+1}(x) - f_{n+1}(x)\, g_n(x) = a_n \neq 0 \qquad \text{where} \quad a_0 a_n > 0.$$

Now consider the Lommel polynomials associated with $f_n(x)$. We have after Palama [4].

$$a_n T_p^{(n)}(x) = f_{n-1}(x)\, g_{n+p}(x) - f_{n+p}(x)\, g_{n-1}(x) \tag{4}$$

and

$$a_n S_p^{(n)}(x) = f_{n+p} g_n(x) - f_n(x)\, g_{n+p}(x) \tag{5}$$

Hence the expression

$$a_n [\, T_{p+1}^{(n)}(x)\, S_p^{(n)}(x) - T_p^{(n)}(x)\, S_{p+1}^{(n)}(x)],$$

after substituting (4) and (5), reduces to $a_{n-1}a_{n+p}$ and by theorem 2, this theorem is established, i.e., the two sequences $\{T_p^{(n)}(x)\}$ and $\{S_p^{(n)}(x)\}$ $p = 0,1,2,3,\dots$ are orthogonal.

The following expressions of type (1) are known for the classical polynomials (see Toscano, [3])

$$L_n^{(\alpha)}(x)\, F_{n+1}^{(\alpha)} - L_{n+1}^{(\alpha)}(x)\, F_n^{(\alpha)}(x) = \frac{\Gamma(\alpha+n+1)}{(n+1)!\,\Gamma(\alpha+1)}$$

$$P_n^{(\alpha)}(x)\, R_{n+1}^{(\alpha)}(x) - P_{n+1}^{(\alpha)}(x)\, R_n^{(\alpha)}(x) = \frac{(2\alpha)_n}{(n+1)!}$$

$$H_n(x)\, G_{n+1}(x) - H_{n+1}(x)\, G_n(x) = n!$$

where $L_n^{(\alpha)}(x)$, $P_n^{(\alpha)}(x)$, and $H_n(x)$ are the Laguerre, the ultrasherical, and the Hermite polynomials respectively, whereas $F_n^{(\alpha)}(x)$, $R_n^{(\alpha)}(x)$, $G_n(x)$ are the polynomials associated with them.

* * * * * * *

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Duke University

ON NATURAL BOUNDARIES OF A GENERALIZED LAMBERT SERIES

Francis Regan and Charles Rust

1. Introduction. In 1932, Feld [1][1] introduced the series

$$F(z) = \sum_{n=1}^{\infty} \frac{a_n b_n z^n}{1 - a_n z^n} \tag{1}$$

and showed that any analytic function can be represented in this form plus a constant. Doyle [2] determined the regions of convergence, expansion in power series and the inversion of power series into a generalized Feld series, which included (1) as a special case.

It is the purpose of this paper to investigate the Feld series for natural boundaries. In establishing natural boundaries for this series, conditions on the sequences $\{a_n\}$ and $\{b_n\}$ of (1) are determined. Before obtaining these conditions three theorems dealing with natural boundaries of the Lambert series

$$L(z) = \sum_{n=1}^{\infty} \frac{c_n z^n}{1 - z^n} \tag{2}$$

are presented and are used in broadening the conditions for natural boundaries for (1).

2. As an immediate consequence of the work of Knopp [3], three theorems dealing with natural boundaries of the Lambert series follow.

Theorem 1. *If $\{c_n\}$ is a null sequence of positive real numbers, then the unit circle is a natural boundary of the Lambert series* (2).

Since $\{c_n\}$ is a null sequence, then

$$\lim_{n\to\infty} \frac{\sum_{r=1}^{n} c_r}{n \sum_{kr<n} c_{kr}} = 0,$$

for a denumerable set of k's. Hence the unit circle is a natural boundary of (2).

We come next to

1. The numbers appearing in [] refer to reference given in bibliography at end of paper.

Theorem 2. *If* $\{c_n\}$ *is a real sequence such that* $0 < c < c_n < C < \infty$, *then the unit circle is a natural boundary of the function represented by the Lambert series* (2).

Using Knopp's method, we let $z_0 = e^{2\pi i \frac{k'}{k}}$, with $(k',k) = 1$, be any rational point on the unit circle and then from the Lambert series, we have

$$\sum_{n=1}^{\infty} \frac{c_n z^n}{1 - z^n} = \sum_{v=1}^{\infty} \frac{c_{kv} z^{kv}}{1 - z^{kv}} + \sum_{k+n} \frac{c_n z^n}{1 - z^n}. \tag{3}$$

Multiply both sides of (3) by $(1 - z/z_0)$ and take the limit as z approaches z_0 radially. We examine the two limits on the right. Hence

$$\lim_{z \to z_0} \left\{ (1 - z/z_0) \sum_v \frac{c_{kv} z^{kv}}{1 - z^{kv}} \right\} \geq$$

$$\frac{c}{k} \lim_{\zeta \to 1} \sum \frac{\zeta^{kv}}{1 + \zeta^k + \dots + \zeta^{(v-1)k}} \tag{4}$$

which, if $\zeta^k = y$, we see that

$$\lim_{z \to z_0} \left\{ (1 - z/z_0) \sum_v \frac{c_{kv} z^{kv}}{1 - z^{kv}} \right\} \geq$$

$$\frac{c}{k} \lim_{y \to 1} \sum \frac{y}{v} = \frac{c}{k} \lim_{y \to 1} \ln \frac{1}{1 - y} \to \infty.$$

On the other hand

$$\left| (1 - z/z_0) \sum_{k+n} \frac{c_n z^n}{1 - z^n} \right| \leq (1 - \zeta) \sum \frac{|c_n| |z^n|}{|1 - z^n|} \tag{5}$$

When n is not a multiple of k, $|1 - z^n| > h > 0$, where $h = 1$ when $k = 2$ and $h = \sin 2\pi/k$ when $k > 2$. Hence

$$\left| (1 - z/z_0) \sum_{k+n} \frac{c_n z^n}{1 - z^n} \right| < \frac{C}{h} (1 - \zeta) \sum \zeta^n < \frac{C}{h} < \infty.$$

Since (4) is unbounded and (5) bounded, we have

$$\lim_{z \to z_0} \left\{ (1 - z/z_0) \sum \frac{c_n z^n}{1 - z^n} \right\} \to \infty$$

Hence the unit circle is a natural boundary of the function represented (2). Lastly, we have

Theorem 3. *If $\{c_n\}$ is a real sequence such that $0 < c_n \leq C < \infty$ and if for a denumerable set of positive integers k, the series $\sum_{v=1}^{\infty} c_{kv}/kv$ is divergent, then the unit circle is natural boundary of the function represented by the series* (2).

As before we consider the two limits in (4) and (5). From (4), we easily get

$$\lim_{\zeta \to 1} \left\{ (1-\zeta) \sum_{v=1}^{\infty} \frac{c_{kv}\zeta^{kv}}{1-\zeta^{kv}} \right\} \geq \frac{1}{k} \lim_{y \to 1} \sum \frac{c_{kv}}{v} y^v \to \infty ,$$

([4], p.177), and it follows easily that the second is bounded. Whenever, this is true for a denumerable set of k's, then the theorem follows.

3. Before discussing the general Feld series, we shall state some theorems which are special cases, the proofs of which are obvious.

Theorem 4. *If $a_n = a^n$ and $b_n = 1$, then the Feld series represents a function which has the circle $|z| = 1/|a|$ as a natural boundary.*

The analog to Knopp's extension of the Franel theorem is

Theorem 5. *If the radius of convergence of $\sum a^n b_n z^n$ is greater than or equal to $1/|a|$, and if the numbers b_n are such that for a certain k, the series $\sum_{k=1}^{\infty} \frac{b_{kv+l}}{kv+l}$ with $1 \neq 0, 1, \ldots, k-1$,* converge *if for such a k a relatively prime integer k', we set $z_0 = (1/a)e^{2\pi i \frac{k}{k}}$, then for radial approach*

$$\lim_{z \to z_0} \left\{ (1 - z/z_0) \sum \frac{a^n b_n z^n}{1 - a^n z^n} \right\} \equiv \sum_{v=1}^{\infty} \frac{b_{kv}}{kv} .$$

If for a denumerably many k $\sum (b_{kv}/kv) \neq 0$, then the circle $|z| = 1/|a|$ is a natural boundary of the function represented by (1).

Theorem 6. *If the radius of convergence of $\sum a^n b_n z^n$ is greater than or equal to $1/|a|$, and the b_n are real and positive, the series $\sum \frac{a^n b_n z^n}{1 - a^n z^n}$ represents a function which has the circle $|z| = 1/|a|$ as a natural boundary when any one of the following conditions on the b_n is satisfied*

(a) *when the numbers* b_n *form a null sequence, or*

(b) *when the numbers* b_n *satisfy the inequalities* $0 < b \leq b_n \leq B < \infty$, *or*

(c) *when the numbers* b_n *satisfy* $0 < b_n \leq B < \infty$ *and the series* $\sum_{v=1}^{\infty} (b_{kv}/kv)$,

diverge for a denumberable set of positive integers k.

Now let us consider the series when $a_n = a$ and $b_n = 1$. Only when $|a| < 1$ will be investigated for if $a = 1$ the Feld series (1) reduces to the Lambert series and if $|a| > 1$, by means of the transformation $az = w$, we obtain $\sum \dfrac{a^{1-n} b_n w^n}{1 - a^{1-n} w^n}$, where $|a^{1-n}| < 1$, when $n > 1$.

Lemma. If $|a| < 1$, then $\sum \dfrac{az^n}{1 - az^n} = \sum \dfrac{a^n z^n}{1 - z^n}$ for $|z| \leq \zeta < 1$.

This lemma is easily established by using Weierstrass theorem on double series.

Theorem 7. *If* $|a| < 1$, *then the series* $\sum \dfrac{az^n}{1 - az^n}$, *represents a function having the unit circle as a natural boundary.*

From the lemma above, we readily obtain

$$\lim_{z \to z_0} \left\{ (1 - z/z_0) \sum \frac{az^n}{1 - az^n} \right\} = \lim_{z \to z_0} \left\{ (1 - z/z_0) \sum \frac{a^n z^n}{1 - z^n} \right\} = \sum \frac{a^{kv}}{kv} \neq 0$$

where z_0 is any rational point on the unit circle. Hence the unit circle is a natural boundary.

Returning to the general Feld series (1), we impose two conditions on the numbers a_n. Firstly, $|a_n| < 1$ and secondly, $\lim_{n \to \infty} \sqrt[n]{|a_n|}$ is 1. Now assuming that the radius of convergence of $\sum a^n b_n z^n$ is greater than or equal to unity, the Feld series is uniformly convergent for $|z| \leq \zeta < 1$. Expending in a power series, we have $\sum \dfrac{a_n b_n z^n}{1 - a_n z^n} = \sum A_n z^n$, where $A_n = \sum_{d/n} a_d^{n/d} b_d$. By means of a generalized Moebius function [2], the sequence $\{a_n b_n\}$ may be expressed in terms of the sequence $\{A_n\}$. This function is defined as follows.

$$S(1) = 1$$

$$\sum_{d/u} a_{(n/u)}^{(u/d)-1} S(d) = 0 \quad \text{for } u > 1.$$

We can show that within the region of convergence of $\sum a_n b_n z^n$ and for $|z| < 1$, $\sum \dfrac{a_n b_n z^n}{1 - a_n z^n} = \sum A_n z^n$, where $a_n b_n = \sum_{d/n} S(n/d) A_d$, as follows. Multiply both sides of $A_d = \sum_{d_1/d} a_{d_1}^{d/d_1} b_{d_1}$, by $S(n/d)$ and summing over all disvisors of n, we obtain

$$\sum_{d/n} S(n/d) A_d = \sum S(n/d) \sum_{d_1/d} a_{d_1}^{d/d^1} b_{d_1} = \sum_{d/n} S(n/d) \sum_{d_1/(n/d)} a_{d_1}^{n/(d_1 d)} b_{d_1}$$

$$= \sum_{d/n} S(d) \sum_{d_1 | \frac{n}{d}} a_{d_1}^{n/(d_1 d)-1} a_{d_1} b_{d_1} = \sum_{d_1 d/n} S(d) a_{d_1}^{n/(d_1 d)-1} a_{d_1} b_{d_1},$$

$$= \sum_{d_1/d} a_{d_1} b_{d_1} \sum_{d | \frac{n}{d_1}} S(d) a_{d_1}^{n/(d_1 d)-1}.$$

But let $n/d = u$, the coefficients of each $a_{d_1} b_{d_1}$ become $\sum_{d/u} S(d) a_{n/u}^{(u/d)-1}$ which from the definition equals zero for all $u = 1$ and equals one when $u = 1$. But when $u = 1$, $n/d_1 = 1$ or $d_1 = n$. Therefore

$$\sum_{d/n} S(n/d)\, A_d = a_n b_n \ .$$

Now, expand the Lambert series (2) in a power series, we get $\sum \dfrac{c_n z^n}{1 - z^n} = \sum A_n z^n$, where $A_n = \sum_{d/n} d_d$.

If we now let this A_n be the coefficients of the power series we obtained by expending the Feld series, we will have $\sum \dfrac{a_n b_n z^n}{1 - a_n z^n} = \sum \dfrac{c_n z^n}{1 - z^n}$, where $a_n b_n = \sum_{d/n} S(n/d) \sum_{d_1/d} c_{d_1}$. We now conclude that if $|c_n| < C < \infty$ and $|a_n| < 1$, then the Feld series and the derived Lambert series represent the same function within the unit circle. Since these two series represent the same function within the unit circle, if we multiply both by the same factor and take the limit as a point is approached radially on the unit circle, we will obtain the same limit. Hence using Knopp's theorem for the Lambert series, we easily eatablish

Theorem 8. *If the numbers* a_n *and* b_n *of the Feld series are such that* $|a_n| < 1$ *and* $a_n b_n = \sum_{d/n} S(n/d) \sum_{d_1/d} c_{d_1}$, *where for an integer k, all k series* $\sum (c_{kv+l}/_{kv+l})$ *for* $1 = 0, 1, \ldots, k-1$, *converge, and if for such a k and a relatively prime k', we set* $z_0 = e^{2\pi i \frac{k'}{k}}$, *then for radial approach*

$$\lim_{z \to z_0} \left\{ (1 - z/z_0) \sum \frac{a_n b_n z^n}{1 - a_n z^n} \right\} = \sum \frac{c_{kv}}{kv}$$

If $\sum \frac{c_{kv}}{kv} \neq 0$ *for a denumerbale set of integral values of k, the unit circle will be a natural boundary for the function represented by the Feld series.*

If we make use of Theorem 2, we conclude

Theorem 9. *If the numbers* a_n *and* b_n *of the Feld series are such that* $|a_n| < 1$ *and* $a_n b_n = \sum_{d/n} S(n/d) \sum_{d_1/d} c_{d_1}$, *where* $0 < c \leq c_n \leq C < \infty$, *then the unit circle is a natural boundary for the function represented by the Feld series.*

From Theorem 3, we have

Theorem 10. *If the numbers* a_n *and* b_n *of the Feld series are such that* $|a_n| < 1$ *and* $a_n b_n = \sum_{d/n} S(n/d) \sum_{d_1/d} c_{d_1}$, *where* $0 < c_n \leq C < \infty$ *and* $\sum \frac{c_{kv}}{kv}$ *is divergent for a denumerable set of k's, then the unit dircle is a natural boundary for the function represented by the Feld series.*

* * * * * * * *

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PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California*

PROPOSALS

313. *Proposed by Sidney Kravitz, East Paterson, New Jersey.*

Show that Christmas falls on a Sunday more often than once every seven years.

314. *Proposed by J.M. Gandhi, Lingraj College, Belgaum, India.*

Show that $2^{n-1} = 2({}_nC_2 - {}_nC_3) + 4({}_nC_4 - {}_nC_5) + 6({}_nC_6 - {}_nC_7) + \dots;$ the last term being $n({}_nC_n)$ when n is even and $-(n-1)_nC_n$ when n is odd.

315. *Proposed by P.D. Thomas, Eglin Air Force Base, Florida.*

Under the restriction $f(c - x) = f(x)$ show that $\int_0^c (a + bx) f(x)\, dx$ may be written $(a + (bc)/2) \int_0^c f(x) dx$.

316. *Proposed by A.K. Rajagopal, Lingraj College, Belgaum, India.*

Prove that $(1 + \cos n\theta)$, n an integer, has a factor $(1 + \cos \theta)$ if and only if n is odd.

317. *Proposed by Ben K. Gold, Los Angeles City College.*

Prove that $(e + x)^{(e-x)} > (e - x)^{(e+x)}$ for $0 < x < e$.

318. *Proposed by Chih-yi Wang, University of Minnesota.*

Evaluate $$\lim_{x\to\infty} \frac{x^{\log x}}{(\log x)^x}$$

319. *Proposed by Barney Bissinger, Lebanon Valley College, Pennsylvania*

Show that $$\frac{\sin n\theta}{\sin \theta} = \cos^{n-1}\theta \left\{1 + \frac{\text{Cos } \theta}{\text{Cos } \theta} + \frac{\text{Cos } 2\theta}{\text{Cos}^2 \theta} + \dots + \frac{\text{Cos}(n-1)\theta}{\text{Cos}^{n-1}\theta}\right\}$$

for those values of θ for which the terms are defined.

SOLUTIONS

Late Solutions

274, 286, 288, 289, 290; *B. Keshava, R. Pai, Mangalore, India*
286, 287, 288; *Gene B. Parrish, Durham, North Carolina*
291; *A.K. Rajagopal, Lingraj College, Belgaum, India*

A Planar Area

292. [January 1957] *Proposed by Eugenio Calabi and Chih-yi Wang, University of Minnesota.*

Find the area of the region in the real xy plane such that

$$|\sinh x \ \sinh y| < 1.$$

I. *Solution by Gene B. Parrish, Burham, North Carolina.* From the symmetry of the area for which $-1 < |\sinh x \sinh y| < 1$, the area integral may be confined to the first quadrant, with x ranging from 0 to ∞ and y ranging from 0 to $\sinh^{-1}\operatorname{csch} x = ln[(e^x + 1)/(e^x - 1)]$. Denoting by A the first-quadrant portion of the area, the total area sought is

$$4\ A = 4 \int_0^\infty ln\ [(e^x + 1)/(e^x - 1)]\ dx$$

Letting $(e^x - 1/(e^x + 1) = u$, then

$$4\ A = -8 \int_0^1 ln\ u\ du/(1 - u^2) = \pi^2$$

where the last integral is given in Table 108 of Bierens de Hans', *Nouvelles Tables D'Integrales Definies* and in Edwards', *Treatise on Integral Calculus*, Vol. II, p. 249.

II. *Solution by Chih-yi Wang.* By symmetry, it suffices to find the area in the first quadrant. Let A be the total area of the required region. By using the relation $\sinh^{-1}u = \log\ (u + \sqrt{1+u^2}\ )$, we have

$$A/4 = \int_0^\infty \sinh^{-1}\ (\operatorname{csch} x)\ dx = \int_0^\infty \log\ (\operatorname{csch}\ x + \coth x)\ dx$$

$$= \int_0^\infty \log \frac{1 + \cosh x}{\sinh x}\ dx = \int_0^\infty \log \frac{e^x + 1}{e^x - 1}\ dx = \pi^2/4$$

(Pierce 520)

Therefore, $A = \pi^2$

III. *Solution by Eugenio Calabi.* Let $x = \sinh^{-1}(\tan u \cos v)$, $y = \sinh^{-1}(\cos u \tan v)$. Since the Jacobian

$$\left| J\frac{(x,y)}{(u,v)} \right| \equiv 1,$$

the area in question is the same as that of $|\sin u \sin v| < 1$, but we know this relation holds for $|u| < \pi/2$, $|v| < \pi/2$. Hence the answer is π^2.

An Euclidean Asymptotic Construction of e

293. [January 1957] *Proposed by Raphael T. Coffman, Richland, Washington.*

Given a line of unit length, construct geometrically a line of length $(1 + 1/n)^n$, where n is an integer.

Solution by Leon Bankoff, Los Angeles, California. Consider the more general problem of constructing geometrically a line of length $(1 + a)^n$, where n is an integer and a is the length of a constructible line segment.

Extend the given segment $AB = 1$ so that $BC = a$ and $AC = 1 + a$. Erect perpendiculars BB' and CC' to AC at B and C respectively. Describe arc (A, AC) cutting BB' at D, and extend AD to cut CC' at E. Then $AE = AD \cdot AC/AB = AC^2$. Now describe arc (A, AE) cutting BB' in F, and extend AF to meet CC' at G. Then $AG = AF \cdot AC/AB = AE \cdot AC = AC^3$. Repetition of this procedure leads in an obvious way to the construction of $(1 + a)^n$ for any desired value of n. For the special case, $(1 + 1/n)^n$, let $a = 1/n$.

As $e = \lim_{n\to\infty} (1 + 1/n)^n$ we have an asymptotic construction of e.

Also solved by Dermott A. Breault, Carnegie Institute of Technology; David J. Cartmell, The College of Wooster, Wooster, Ohio; Howard Eves, University of Maine; B. Keshava R. Pai, Mangalore, India; Chih-yi Wang, University of Minnesota and the proposer.

Concyclic Nine Point Centers

294. [January 1957] *Proposed by N.A. Court, University of Oklahoma.*

The nine-point centers of the four triangles formed by four concyclic points taken three at a time lie on a circle.

I. *Solution by Sister M. Stephanie, Georgian Court College, New Jersey.* Let the circle on which the four points lie be a unit circle, and the four points be, in a system of complex coordinates, t_1, t_2, t_3 and t_4. The nine-point circle of triangle $t_1t_2t_3$ will then be $z = s_1/2 + t/2$ where $s_1 = t_1 + t_2 + t_3$. Its center is at $s_1/2$ and its radius in one-half the radius of the unit circle. The nine-point circles of the other triangles are found similarly.

Consider the circle $z = S_1/2 + t/2$ where $S_1 = t_1 + t_2 + t_3 + t_4$. This circle is the locus of the centers of the four nine-point circles, for the substitution of a properly chosen unit vector, e.g., $-t_4$, yields each of the centers in turn. It has center at $S_1/2$ and radius equal to one-half the unit circle.

Note: The point $S\ /2$ is the point of intersection of the four nine-point circles since it lies on each; it is also the center of the equilateral hyperbola which can be drawn through the four given points.

II. *Solution by B. Keshava R. Pai, Mangalore, India.* The radius of the nine-point circle of a triangle is equal to half of the circumradius of the triangle. Therefore, the radii of the four nine-point circles of the four triangles are equal, since the four points are concyclic and so the circumcircle, is one and the same, viz the circle passing through the four points.

Now, the centre locus of a family of Rectangular hyperbolas passing through the vertices of a triangle is the nine-point circle of the triangle. Consider each of the four triangles separately. The four nine-point circles are the four centre loci of the four sets of Rectangular hyperbolas. Now, through any four points there passes a Rectangular hyperbola. The Rectangular hyperbola passing through the four given points belongs to all the four sets and hence its centre lies on all the four loci. Hence, the four loci, i.e., the four nine-point circles pass through a point.

Since the four nine-point circles have common radii and pass through a common point, the centres of them lie on a circle with the common point as the centre and half the circumradius as radius.

Also solved by J.W. Clawson, Collegeville, Pennsylvania; Huseyin Demir, Kandilli, Eregli, Kdz, Turkey; Howard Eves, University of Maine; A.K. Rajagopal, Lingraj College, Belgaum, India; Chih-yi Wang, University of Minnesota and the proposer.

Radix Identities

296. January 1957 *Proposed by P.A. Piza, San Juan, Puerto Rico.*

Prove that the following equalities

$$385 + 439 + 547 = 367 + 475 + 529$$

$$385^2 + 439^2 + 547^2 = 367^2 + 475^2 + 529^2$$

are true not only when the six distinct 3-digit numbers are considered to belong to the decimal system of numeration, but also when they are regarded as belonging to any system or scale of numerical notation with any base greater than ten.

Solution by James A. Painter, I.B.M. Corporation, Endicott, New York. A three digit number abc in any base, say r, represents $ar^2 + br + c$. Hence the problem is to show:

$$(1)\qquad \begin{aligned}&(3r^2 + 8r + 5) + (4r^2 + 3r + 9) + (5r^2 + 4r + 7) =\\ &(3r^2 + 6r + 7) + (4r^2 + 7r + 5) + (5r^2 + 2r + 9)\end{aligned}$$

$$(2)\qquad \begin{aligned}&(3r^2 + 8r + 5)^2 + (4r^2 + 3r + 0)^2 + (5r^2 + 4r + 7)^2 =\\ &(3r^2 + 6r + 7)^2 + (4r^2 + 7r + 5)^2 + (5r^2 + 2r + 9)^2\end{aligned}$$

for $r \geq 10$.

Collecting like powers of r gives 3 equations from equation (1) and 5 from equation (2).

$5 + 9 + 7 = 7 + 5 + 9$

$(8 + 3 + 4)r = (6 + 7 + 2)r$

$(3 + 4 + 5)r^2 = (3 + 4 + 5)r^2$

$25 + 81 + 49 = 49 + 25 + 81$

$(56 + 54 + 80)r = (84 + 70 + 36)r$

$(64 + 30 + 9 + 72 + 16 + 70)r^2 = (36 + 49 + 4 + 42 + 40 + 90)r^2$

$(40 + 24 + 48)r^3 = (36 + 56 + 20)r^3$

$(25 + 16 + 9)r^4 = (9 + 16 + 25)r^4$

for $r \geq 10$.

Obviously this set of equations hold for any r. The base of a positional notation system must be greater than the value of any symbol used in the system. Since 9 is used in equation (1) and (2), $r > 9$. That is $r \geq 10$.

Also solved by Howard Eves, University of Maine; Harry M. Gehman, University of Buffalo; Erich Michalup, Caracas, Venezuela; Wahin Ng, San Francisco, California; B. Keshava R. Pai, Mangalore, India; Chih-yi Wang, University of Minnesota and the proposer.

Construction of an Equilateral Triangle

297. [January 1957] *Proposed by Dewey Duncan, East Los Angeles Junior College.*

Given a point, a line and a circle in a plane. Construct an equilateral triangle having a vertex on each of them. Determine the criterion for the existence of such a triangle.

Solution by Howard Eves, University of Maine. Designate the given point, line, and circle by P, L, and C. Let L' be obtained by rotating L about P through an angle of $\pm 60°$. Suppose L' cuts C in a point Q. Then the perpendicular bisector of PQ cuts L in R such that triangle PQR is equilateral.

A necessary and sufficient condition for a solution is that L' intersects C. Because the angle of rotation may be $\pm 60°$, there may be as many as 4 solutions to the problem. It is easy to construct examples showing exactly 3, 2, 1, and no solutions.

Also solved by J.W. Clawson, Collegeville, Pennsylvania; Huseyin Demir, Kandilli, Eregli, Kdz, Turkey; H.M. Gandhi, Lingraj College, Belgaum, India and the proposer.

An Invariant Curve

298. [January 1957] *Proposed by Huseyin Demir, Kandilli, Bolgesi, Turkey.*

Let $y = f(x)$ be a curve with the following properties

a) $f(x) = f(-x)$

b) $f'(x) > 0$ for $x > 0$

c) $f''(x) = 0$

Determine the weight per unit length $w(x)$ at the point (x, y) such that when the curve is suspended under gravity by any two points on it, the curve will keep its original shape.

Solution by K.L. Cappel, Philadelphia, Pennsylvania. Assume the curve to be suspended at two arbitrary points A and B. Let the weight between A and the y intercept of the curve be W. Then at A, the tension in the curve can be resolved into vertical and horizontal components so that $W/H = \tan\theta$ or $W = H\ dy/dx$.

Now assume the right point of support to be moved from A to A'. If the curve is to retain its shape, there must be no change in the forces at A. This can only be the case if H is a constant. If ds is the length of the segment AA', and dW is its weight, then the weight per unit length will be

$$W_x = \frac{dW}{ds} = \frac{d\ W}{\sqrt{1 + (\frac{dy}{dx})^2}\ dx} \qquad or, \qquad W_x = H \cdot \frac{d^2y/dx^2}{\sqrt{1 + (\frac{dy}{dx})^2}}$$

which can be satisfied by any curve obeying the given conditions.

This problem is analagous to the problem of finding the optimum shape of a masonry arch, when the material of the arch is the only load to be supported, and it is desired to have the thrust load act along the neutral axis in order to eliminate bending moments.

Also solved by the proposer

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 204. If A, B, C and D are vectors and $[ABC]$ is a scalar triple product, prove that

$$[BCD]\ A - [CDA]\ B + [DAB]\ C - [ABC]\ D = 0$$

[*Submitted by M.S. Klamkin.*]

Q 205. An airplane with an airspeed of 100 mph, flying into the wind, passes over a point just as a lighter than air balloon is released. After a while the plane turns down-wind and overtakes the balloon 8 miles from the point of release and one half hour after it was released. Assuming constant wind and no time lost in turning, how far did the plane fly before turning? [*Submitted by Richard K. Guy.*]

Q 206. If x, y and z are positive and if $x + y + z = 1$ prove that

$$(1/x - 1)(1/y - 1)(1/z - 1) \geq 8$$ [*Submitted by M.S. Klamkin*]

Q 207. Find the Arithmetic Progression for which the nth term + mth = $(n + m)$th term. [*Submitted by B. Keshava R. Pai.*]

Q 208. If $f(x) \equiv f(x + 1) \equiv f(x + \sqrt{2})$ and $f(0) = \sqrt{2}$, find $f(x)$.
[*Submitted by M.S. Klamkin*]

Q 209. Find the area between $y = x^3$ and $y^2 = 32x$.
[*Submitted by John M. Howell*]

Q 210. If $f(x)$ can be integrated in finite form, show that the inverse function $f^{-1}(x)$ can also be integrated in finite form.
[*Submitted by M.S. Klamkin*]

ANSWERS

A 204 Let

$$I = \begin{vmatrix} A & B & C & D \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

(Continued on back of Contents)

BIOGRAPHICAL DATA -

Biographical data will not be published unless there is a demand for it. This is due to pressure for the publication of material which seems to be of greater interest to our readers.

Editor.

EDITORIAL NOTE -

Readers are invited to send us their selections of remarkable situations in Mathematics that can be explained at the freshman level.

H.V. Craig, Editor.

(Answers, continued from page 57)

Then $I \cdot i = I \cdot j = I \cdot k = 0$ so $I = 0$. Expanding I by minors, we get the desired result.

A 205. The speed of the wind is 16 mph. Apply this to the whole problem bringing the balloon and air to rest. Therefore, the plane flies ¼ hour before turning, or 25 miles relative to the air or 21 miles relative to the ground.

A 206. The sum of the three factors on the left hand side of the inequality equals one. Thus, their minimum occurs when $x = y = z = 1/3$. Hence, the inequality follows.

A 207. In the progression $a,\ a + d,\ a + 2d, \ldots$ substituting the values of the nth, mth and $(n + m)$th terms in the given condition leads to the result $a = d$. Hence, the progression is $a,\ 2a,\ 3a, \ldots$

A 208. Since $f(x)$ has two independent periods, it must be the constant $f(x) \equiv \sqrt{2}$

A 209. By inspection the curves intersect at $(0,0)$ and $(2,8)$. Since the area inside a parabola and a cubic are $1/3$ and $1/4$ respectively of the circumscribing rectangle, the area between the curves is $(\frac{5}{12})\ (16) = \frac{20}{3}$.

A 210. $I = \int f^{-1}(x)dx$. Let $y = f^{-1}(x)$, then $x = f(y)$ and $dx = f'(y)dy$. Then, $I = \int y\ f'(y)dy = y\ f(y) - \int f(y)\ dy$.

From the Preface to:

"THE TREE OF MATHEMATICS"

There is a great deal being published *about mathematics* these days, and that is fine; but this book *is mathematics* in the sense that it presents the epitomes of the main branches of the subject beginning with high school algebra and extending far into graduate work.

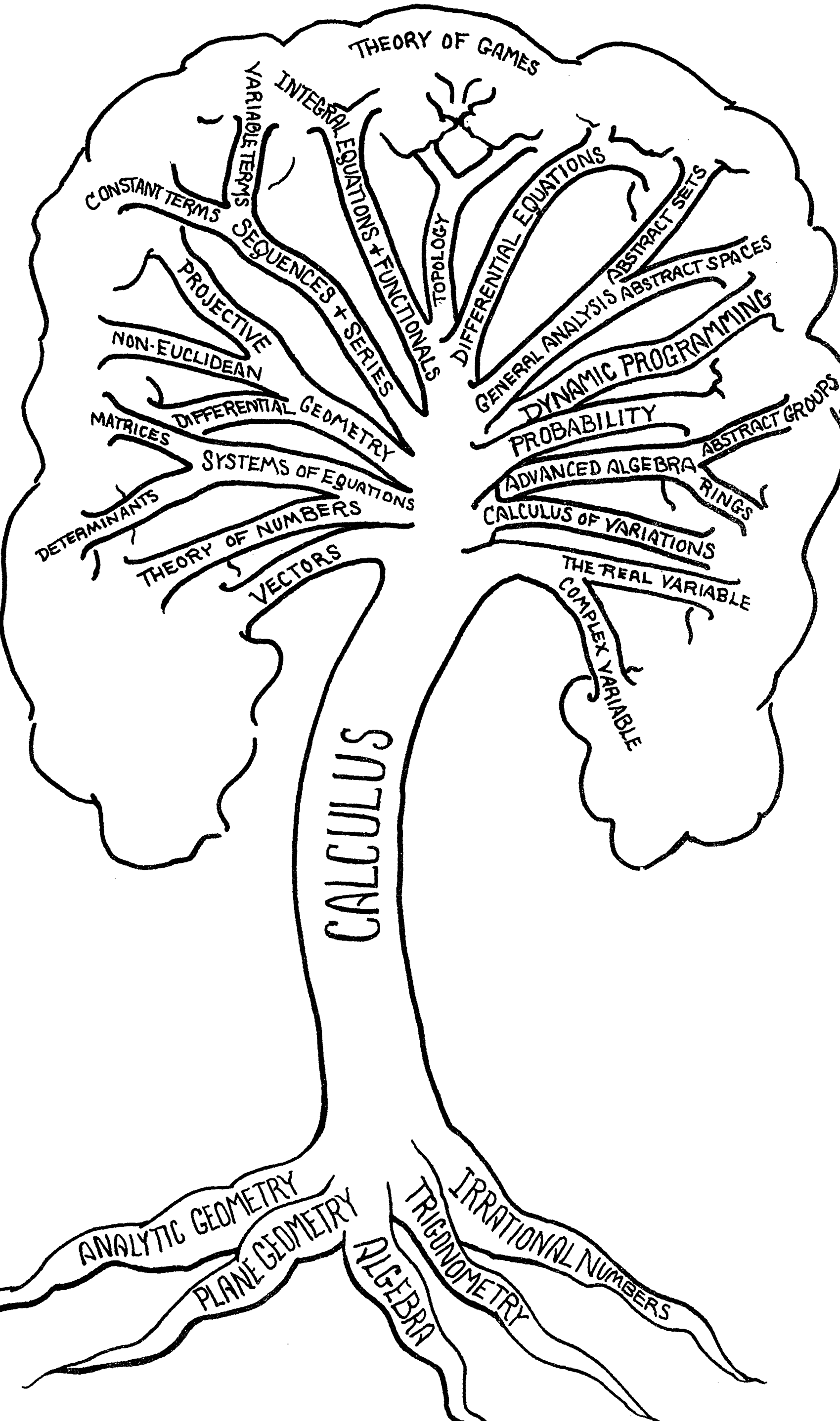
When writing this treatise the authors gave great attention to making it both meaningful and understandable, an art in which most of them are pastmasters. The practice was to start on ground familiar to everyone and construct a highway, free from road blocks, through the wonderful world of mathematics.

This book is, for the most part, a response to requests for source material from two classes of people: those who need an ever increasing knowledge of mathematics in their jobs, such as engineers; and others who have gone little if any beyond arithmetic and either need more mathematics or just want to know "what it is all about." Inherent in a satisfactory response to these requests is an answer to the needs of teachers and students of mathematics and related subjects who desire to extend their horizons by home-study, and to do so in a minimum of time.

In terms of class-work, the first seven chapters and perhaps selections from later ones would constitute about a one semester survey course in high school, while by passing swiftly over the first three or four chapters the entire volume could be covered in a two semester survey course in college.

But classwork is always slower than effective individual study. The latter could reduce the above periods to months or possibly weeks by study at home during the evenings.

Due to its broad coverage, this treatise offers an excellent opportunity for "reading up" on specific topics. By looking a topic up in the index you will find references to the page or pages where it is discussed or its meaning depicted by usuage. Of course, a similar procedure is possible with textbooks, but it would require some two dozen of them to cover all the topics in this treatise.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